Versatile Mathematics

COMMON MATHEMATICAL APPLICATIONS

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Attributions This book benefited tremendously from others who went before and freely shared their creative work. The following is a short list of those whom we have to thank for their work and their generosity in contributing to the free and open sharing of knowledge.

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Chapter

4 Probability

It is often necessary to “guess” about the outcome of an event in order to make a decision. Politicians study polls to guess their likelihood of winning an election. Teachers choose a particular course of study based on what they think students can comprehend. Doctors choose the treatments needed for various diseases based on their assessment of likely results. You may have visited a casino where people play games chosen because of the belief that the likelihood of winning is good. You may have chosen your course of study based on the probable availability of jobs.

You have, more than likely, used probability. In fact, you probably have an intuitive sense of probability. Probability deals with the chance of an event occurring. Whenever you weigh the odds of whether or not to do your homework or to study for an exam, you are using probability. In this chapter, you will learn how to solve probability problems using a systematic approach.

SECTION 4.1 Basic Concepts of Probability

Before learning basic concepts of probability, there is some terminology that we need to get familiar with. An **experiment** is a planned operation carried out under controlled conditions. If the result is not predetermined, then the experiment is said to be a chance experiment. Rolling one fair, six-sided die twice is an example of an experiment. A result of an experiment is called an **outcome**. The **sample space** of an experiment is the set of all possible outcomes. We can represent a sample space in three possible ways:

1. List all possible outcomes. For example, if you roll a six-sided die (the standard die that we'll use for our examples), the sample space S could be written
$$S = \{1, 2, 3, 4, 5, 6\}$$
2. Create a tree diagram, showing different ways that events in order could happen.
3. Draw a Venn diagram (we'll see this later in the chapter).

An **event** is any combination of outcomes. Upper case letters like A and B represent events. For example, if the experiment is to flip one fair coin three time, event A might be getting at most one head. The probability of an event A is written $P(A)$.

EXAMPLE 1 TWO SIBLINGS

Consider randomly selecting a family with 2 children where the order in which different gender siblings are born is significant. That is, a family with a younger girl and an older boy is different from a family with an older girl and a younger boy. What would the sample space look like?

Solution

If we let G denote a girl, B denote a boy, then we have the following:
$$S = \boxed{\{GG, BB, GB, BG\}}$$
This notation represents families with 2 girls, 2 boys, an older girl and a younger boy, an older boy and a younger girl.

TRY IT

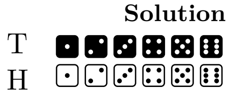
What would the sample space S look like if we considered a family with 3 children? Remember, the order of children born is significant.

The following example describes a familiar experiment that can actually be easily performed.

EXAMPLE 2 TOSSING A COIN AND ROLLING A DIE

Suppose we toss a fair coin and then roll a six-sided die once. Describe the sample space S.

Solution



Let T denote Tails, and H denote Heads. Then:
$$S = \boxed{\{T1, T2, T3, T4, T5, T6, H1, H2, H3, H4, H5, H6\}}$$

TRY IT

A large bag contains red, yellow, blue, and green marbles. Describe the sample space S of an experiment when 2 marbles are selected at random, one at a time, and the order of selection is significant.

Now that we have covered basic terminology, it is time to define a few terms and rules:

Probability

Probability is a measure that is associated with how certain we are of outcomes of a particular experiment or activity.

It is defined as the proportion of times we would expect a particular outcome to occur if we repeated the experiment many times.

The basic rules of probability are:

1. $0 \leq P(A) \leq 1$ for any event A; that is, all probabilities are between 0 and 1
2. $P(A) = 0$ means that event A will not occur
3. $P(A) = 1$ means that event A is certain to occur
4. $P(E_1) + P(E_2) + \cdots + P(E_n) = 1$; that is, the sum of probabilities of all possible n outcomes of an experiment E is 1

Often we use percentages to represent probabilities. For example, a weather forecast might say that there is 85% chance of rain in Frederick tomorrow. Or there is 67% chance that the Baltimore Orioles will win their next series. Or a particular poker player has a 35% chance of winning the game with his current hand. As you might have already guessed, 100% chance corresponds to 1, and 0% corresponds to 0.

Theoretical Probability

There are two types of probability: **theoretical** and **empirical**. Theoretical probability is used when the set of all equally-likely outcomes is known. To compute the theoretical probability of an event E, denoted $P(E)$, we use the formula below:

Theoretical probability

$$P(E) = \frac{\text{number of ways } E \text{ can occur}}{\text{total number of possible outcomes}}$$

This makes sense with the definition of probability, namely that it is the proportion of times we would expect E to occur if we repeated the experiment many times.

In the example below, one could probably find the probability by intuition, but it's good to know how to apply the formula, even in what seems to be a simple experiment.

ROLLING A DIE **EXAMPLE 3**

Assume you are rolling a fair six-sided die. What is the probability of rolling an odd number?

Solution

Since half of the numbers on a die are even, and half are odd, intuitively you know that there is 50% chance of rolling an odd number. But how would you compute this probability using the formula above?

There are 6 possible outcomes when rolling a die: 1, 2, 3, 4, 5, and 6. Three of these outcomes are odd numbers: 1, 3, and 5. Let O denote an event when an odd number is rolled. Then

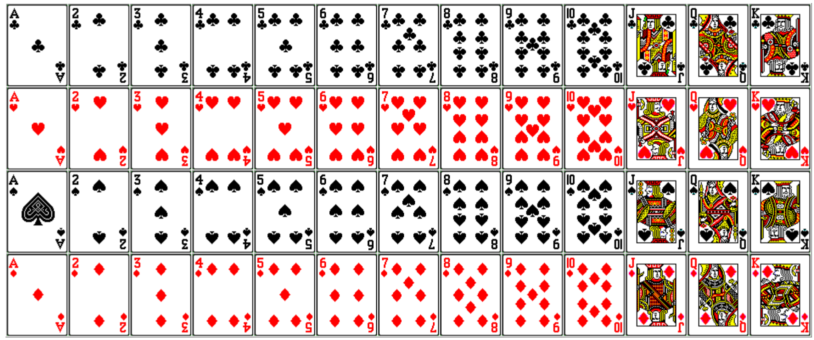
$$P(O) = \frac{3}{6} = \boxed{\frac{1}{2}}$$

TRY IT

When rolling a fair six-sided die, what is the probability of rolling a number less than 2? Greater than 6?

Often, it is not necessary to actually list all the possible outcomes, as long as you can determine how many outcomes there are.

The example below uses cards to calculate probabilities, so in case you are not familiar with a standard deck of 52 cards, the diagram below may be helpful.



There are four *suits* (arranged in rows above): clubs, hearts, spades, and diamonds. Clubs and spades are black; hearts and diamonds are red. Each suit contains 13 cards: one Ace, nine numbered cards from 2 to 10, and three *face cards*, so named because they have a drawing of a person. The face cards are Jacks, Queens, and Kings.

EXAMPLE 4 DRAWING A CARD

Suppose you draw one card from a standard 52-card deck. What is the probability of drawing an Ace?

Solution There are 4 aces in a deck of cards. Let A denote an event that the drawn card is an Ace. Then

$$P(A) = \boxed{\frac{4}{52}} = \frac{1}{13}$$

Note: There is no need to simplify the fractions in the answers in this section. In fact, it is often easier to understand an answer in unsimplified form, because we can recognize that the numerator is the number of ways our event can occur, and the denominator is the number of possible outcomes in the sample space.

TRY IT When drawing a card from a standard 52-card deck, what is the probability of drawing a face card? What about the probability of drawing the King of Hearts?

EXAMPLE 5 COOKIES

Lisa's cookie jar contains the following: 5 peanut butter, 10 oatmeal raisin, 12 chocolate chip, and 8 sugar cookies. If Lisa selects one cookie, what is the probability she gets a peanut butter cookie?

Solution The total number of cookies in the jar is 35. Let PB denote the event when a peanut butter cookie is selected, then

$$P(PB) = \boxed{\frac{5}{35}} = \frac{1}{7}$$

TRY IT What is the probability that Lisa gets an oatmeal raisin cookie? What flavor of cookie is Lisa *most likely* to get and why?

Empirical Probability

As long as we can list–or at least count–the sample space and the number of outcomes that correspond to our event, we can calculate basic probabilities by dividing, as we have done so far. But there are many situations where this isn't feasible.

For instance, take the example of a batter coming to the plate in a baseball game. There's no way to even begin to list all the possible outcomes that could occur, much less count how many of them correspond to the batter getting a hit. We'd still like to be able to estimate the likelihood of the batter getting a hit during this at-bat, though. Just as sports fans do, then, we look at this batter's previous performance; if he's gotten a hit in 200 of his last 1000 at-bats, we assume that the probability of a hit this time is $200/1000 = 0.200$.

Empirical probability is used when we observe the number of occurrences of an event. It is used to calculate probabilities based on the *real data* that we observed and collected. To compute the empirical probability of an event E, denoted $P(E)$, we use the formula below:

Empirical probability

$$P(E) = \frac{\text{observed number of times } E \text{ occurs}}{\text{total number of observed occurrences}}$$

This can also be used to answer questions about sampling randomly from a population if we know the breakdown of the group.

FCC STUDENTS **EXAMPLE 6**

Consider the following information about FCC students' enrollment:

Gender	Enrollment
Female	3653
Male	2580

If one person is randomly selected from all students at FCC, what is the probability of selecting a male student?

Solution

The total enrollment is 6233 students, thus we get:

$$P(M) = \boxed{\frac{2580}{6233} \approx 0.414}$$

RESIDENCY **EXAMPLE 7**

Consider the following information about students at Frederick Community College:

Residence	Enrollment
Frederick County	5847
From another county in Maryland	245
Out of state	141

Each student can be assigned to one category only. If one person is randomly selected from the total group, what is the probability this student is from another county in Maryland?

Solution

Adding all the enrollments, we find that the total number of students is 6233, thus

$$P(A) = \boxed{\frac{245}{6233} \approx 0.039}$$

TRY IT

Consider the following information about FCC students' enrollment:

Status	Enrollment
Full-time	2359
Part-time	3874

If one person is randomly selected from the group, what is the probability of choosing a full-time student? Round your answer to 3 decimal places.

The next example contains a two-way table, often referred to as a **contingency table**, which breaks down information about a group based on two criteria. For example, the table below breaks down a group of 130 FCC students based on gender and which hand is their dominant hand:

Gender	Right-handed	Left-handed
Female	58	13
Male	47	12

In order to use this to calculate probabilities if we randomly select someone from the group, we need to calculate totals for each category: the number of males, the number of females, the number of left-handed people, and the number of right-handed people. This is done by simply summing each row and column; if we do that, we obtain the completed table below.

Gender	Right-handed	Left-handed	Total
Female	58	13	71
Male	47	12	59
Total	105	25	130

Notice carefully that these row and column totals do not need to be given, since we can quickly calculate them, but they will often be shown for convenience.

EXAMPLE 8 FCC STUDENTS

Consider the following information about a group of 130 FCC students:

Gender	Right-handed	Left-handed	Total
Female	58	13	71
Male	47	12	59
Total	105	25	130

If one person is randomly selected from the group, what is the probability this student is left-handed?

Solution The probability of selecting a left-handed student is equal to

$$\frac{\text{the total number of left-handed students}}{\text{the total number of all students}}$$

To find the number of left-handed students, you can add up the female and male students who are left-handed, or simply read the total value at the bottom of that column: the answer is 25. Similarly, to find the total number of students, you can add up the values in the four cells at the center of the table, or simply read the total in the bottom right-hand corner: 130.

$$P(L) = \boxed{\frac{25}{130} \approx 0.192}$$

TRY IT

Using the table above, find the probability of selecting a female student. Round your final answer to three decimal places.

Now, let's go back to example 3: rolling a six-sided die and considering the event of rolling an odd number. If you were to roll the die only a few times, you might be surprised if your observed results did not match the probability. If you were to roll the die a very large number of times, you would expect that, overall, 1/2 of the rolls would result in an outcome of "odd number." However, you would not expect exactly 1/2. The long-term relative frequency of obtaining this result would approach the theoretical probability of 1/2 as the number of repetitions grows larger and larger.

This important characteristic of probability experiments is known as the **Law of Large Numbers** which states that as the number of repetitions of an experiment is increased, the empirical probability obtained in the experiment tends to become closer and closer to the theoretical probability.

DEMONSTRATING THE LAW OF LARGE NUMBERS — EXAMPLE 9

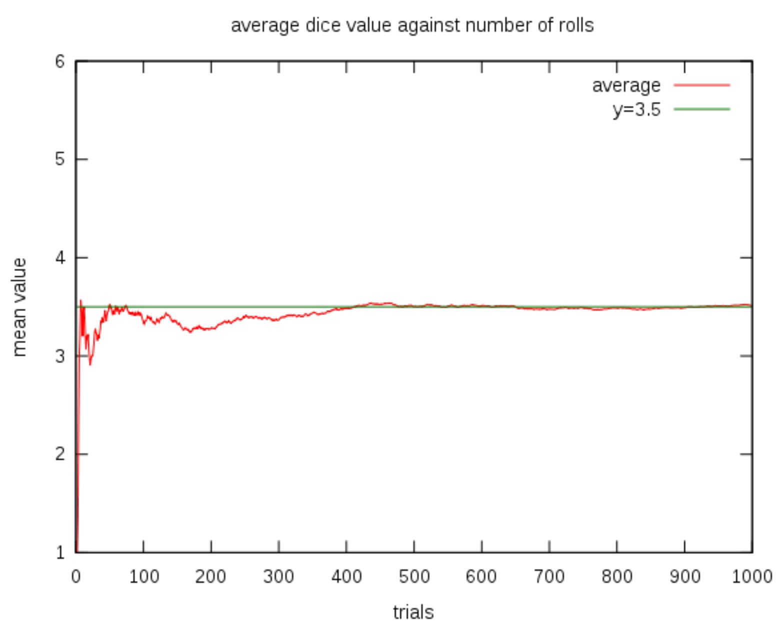
Consider the following experiment: rolling a six-sided die 10 times, recording the outcomes and then taking the average. One can do this experiment by using a random number generator rather than physically rolling a die over and over again. Suppose the results are recorded in a table below:

Roll	1	2	3	4	5	6	7	8	9	10
Outcome	4	2	1	6	2	4	3	2	5	4

We compute the average of the outcomes:

$$\frac{4+2+1+6+2+4+3+2+5+4}{10} = \frac{33}{10} = 3.3$$

What will happen if we roll the die 100 times? 1000 times?



You can see that as the number of rolls in this experiment increases, the average of the values of all the results approaches 3.5. If different people tried doing this experiment, their graphs would show a different shape over a small number of throws (at the left), but over a large number of rolls (to the right) they would be extremely similar.

Exercises 4.1

1. A fair die is rolled. Find the probability of getting 4.

2. A fair die is rolled. Find the probability of getting less than 3.

3. A fair die is rolled. Find the probability of getting at least 5.

4. You have a bag with 20 cherries, 14 sweet and 6 sour. If you pick a cherry at random, what is the probability that it will be sweet?

5. A ball is drawn randomly from a jar that contains 6 red balls, 2 white balls, and 5 yellow balls. Find the probability of drawing a white ball.

6. Suppose you write each letter of the alphabet on a different slip of paper and put the slips into a hat. What is the probability of drawing one slip of paper from the hat at random and getting a consonant? Note that a consonant is any letter that is not a vowel (*a, e, i, o,* or *u*).

7. In a survey, 205 people indicated they prefer cats, 160 indicated they prefer dogs, and 40 indicated they don't enjoy either pet. Find the probability that if a person is chosen at random, they prefer cats.

8. A group of people were asked if they had run a red light in the last year. 150 responded "yes" and 185 responded "no." Find the probability that if a person is chosen at random, they have run a red light in the last year.

9. A U.S. roulette wheel has 38 pockets: 1 through 36, 0, and 00. 18 are black, 18 are red, and 2 are green. A play has a dealer spin the wheel and a small ball in opposite directions. As the ball slows to stop, it can land with equal probability on the 38 slots. Find the probability of the ball landing on green.

10. A glass jar contains 6 red, 5 green, 8 blue and 3 yellow marbles. If a single marble is chosen at random from the jar, what is the probability of choosing

(a) a red marble?

(b) a green marble?

(c) a blue marble?

11. Lisa has a large bag of coins. After counting the coins, she recorded the counts in the table below. She then decided to draw some coins at random, replacing each coin before the next draw.

Quarters	**Nickels**	**Dimes**	**Pennies**
27	18	34	21

(a) What is the probability that Lisa obtains a quarter on the first draw?

(b) What is the probability that Lisa obtains a penny or a dime on the second draw?

(c) What is the probability that Lisa obtains at most 10 cents worth of money on the third draw?

(d) What is the probability that Lisa does not get a nickel on the fourth draw?

(e) What is the probability that Lisa obtains at least 10 cents worth of money on the fifth draw?

12. Suppose you roll a pair of six-sided dice.

(a) List all possible outcomes of this experiment.

(b) What is the probability that the sum of the numbers on your dice is exactly 6?

(c) What is the probability that the sum of the numbers on your dice is at most 4?

(d) What is the probability that the sum of the numbers on your dice is at least 9?

13. I asked my Facebook friends to complete a two-question survey. They answered the following questions: Which beverage do you prefer in the morning: coffee or tea? What is your gender? I summarized the results in following table:

	Coffee	**Tea**	**Total**
Female	37	24	**61**
Male	22	31	**53**
Total	**59**	**55**	**114**

(a) What is the probability that I select a friend who prefers coffee?

(b) What is the probability that I select a friend who is female?

(c) What is the probability that I select a friend who is male and prefers tea?

14. A poll was taken of 14,056 working adults aged 40-70 to determine their level of education. The participants were classified by sex and by level of education. The results were as follows.

Education Level	**Male**	**Female**	**Total**
High School or Less	3141	2434	5575
Bachelor's Degree	3619	3761	7380
Master's Degree	534	472	1006
Ph.D.	52	43	95
Total	7346	6710	14,056

A person is selected at random. Compute the following probabilities:

(a) The probability that the selected person is a male

(b) The probability that the selected person does not have a Ph.D.

(c) The probability that the selected person has a Master's degree

(d) The probability that the selected person is female and has a Master's degree

SECTION 4.2 The Addition Rule and the Rule of Complements

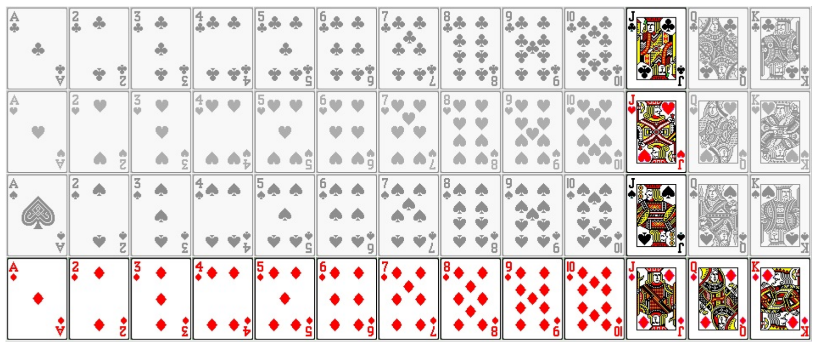
In this section, we will focus on computing probabilities of events involving "or" as well as learning the concept of mutually exclusive events. We will also discuss complementary events and their probabilities.

To start, recall the experiment of drawing one card from a standard deck of cards. Let J denote drawing a Jack, and Q denote drawing a Queen. What is the probability of drawing a Jack? It is, of course, 4/52, and the same goes for the probability of drawing a Queen. Now, what is the probability of drawing a Jack *OR* Queen? By looking back at the deck of cards, we can see that there are 8 cards that are either Jacks or Queens, so

$$P(J\ OR\ Q) = \frac{8}{52},$$

which happens to be the sum of their individual probabilities.

What about, though, if we wanted to find the probability of drawing a Jack or a diamond? Could we just add their individual probabilities (4/52 and 13/52, respectively)? Let's check by looking back at the cards and see which correspond to Jacks or diamonds.



Notice that there are 16 cards that match that description, so the probability is 16/32, which ISN'T the sum of the individual probabilities. What went wrong?

Mutually Exclusive Events

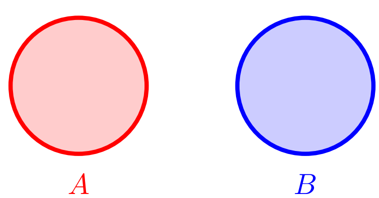
The answer can be found by looking at the diagram above. Notice that if we add up the number of Jacks and the number of diamonds (for a total of 17), we *double count* the Jack of diamonds. This brings us to an important definition that determines how we find the probability of one event OR another occurring: we need to find whether the events are **mutually exclusive** or **disjoint**. That is, can these two events happen at the same time?

Disjoint (Mutually Exclusive) Outcomes

Two outcomes are called **disjoint** or **mutually exclusive** if they cannot both happen at the same time.

Can we draw a card that is both Jack and Queen? Clearly, there is no such card, therefore these events are disjoint. Another familiar example of disjoint events would be getting an even number and getting an odd number when rolling a die. Each number is either even or odd, so these two events are also mutually exclusive. Above, though, we showed that drawing a Jack and drawing a diamond are NOT mutually exclusive, since you can draw the Jack of diamonds.

Notice that the terms **disjoint** and **mutually exclusive** are equivalent and interchangeable. The Venn diagram below illustrates the concept of mutually exclusive events: two events A and B do not overlap; they are disjoint.



Before we formally define a formula for computing probabilities of disjoint events, let us solve some problems by using the rules we already know.

ROLLING A DIE — EXAMPLE 1

Suppose you roll a fair six-sided die once. What is the probability of rolling a 6 or an odd number?

Solution

Since 6 is even, these two events are disjoint; this means that we can simply add the probabilities for rolling a 6 and rolling an odd number:

$$\begin{aligned} P(O \text{ or } 6) &= P(O) + P(6) \\ &= \frac{3}{6} + \frac{1}{6} \\ &= \boxed{\frac{4}{6} = \frac{2}{3} \approx 0.667} \end{aligned}$$

Notice that we could also solve the problem above by examining the sample space and counting all the numbers that fit the description "6 or odd." You'll find that there are often multiple ways to solve probability problems.

TRY IT

If you roll a fair six-sided die once, what is the probability of getting a 5 or a number less than 2?

Here's the formula for mutually exclusive, or disjoint, events:

Addition rule for mutually exclusive events

If A_1 and A_2 are mutually exclusive events, then

$$P(A_1 \text{ or } A_2) = P(A_1) + P(A_2)$$

Furthermore, we can generalize this rule for finitely many disjoint events (where n is the number of events):

$$P(A_1 \text{ or } A_2 \ldots \text{ or } A_n) = P(A_1) + P(A_2) + \cdots + P(A_n)$$

DRAWING A CARD — EXAMPLE 2

Suppose you draw one card from a standard 52-card deck. What is the probability that you get an Ace or a face card?

Solution

There are 4 Aces and 12 face cards in a standard deck of cards. These outcomes are disjoint, since only one card is drawn and Aces and face cards are distinct, so we find the probability as follows:

$$\begin{aligned} P(A \text{ or } F) &= P(A) + P(F) \\ &= \frac{4}{52} + \frac{12}{52} \\ &= \boxed{\frac{16}{52} = \frac{4}{13} \approx 0.308} \end{aligned}$$

TRY IT

If one card is randomly selected from a deck, what is the probability of getting a number or a red Jack? If one card is randomly selected from a deck, what is the probability of selecting a red suit or a black suit?

EXAMPLE 3 MARBLES

A large bag contains 28 marbles: 7 are blue, 8 are yellow, 3 are white, and 10 are red.

(a) If one marble is randomly selected, what is the probability that it is either red or yellow?

(b) If one marble is randomly selected, what is the probability that it is neither white nor red?

Solution

(a) Clearly, selecting a red or yellow marble are disjoint events, so we find the answer by adding together the individual probabilities:

$$\begin{aligned} P(R \text{ or } Y) &= P(R) + P(Y) \\ &= \frac{10}{28} + \frac{8}{28} \\ &= \boxed{\frac{18}{28} = \frac{9}{14} \approx 0.643} \end{aligned}$$

(b) A question involving "neither/nor" is different from a question involving "either/or", because NOT white and NOT red are not mutually exclusive events. Thus, to answer this question, we will simply count how many marbles are not white and also not red. This includes the blue and yellow marbles, for a total of $7 + 8 = 15$:

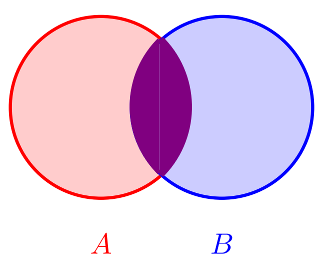
$$P(\text{not } W \text{ and not } R) = \boxed{\frac{15}{28} \approx 0.54}$$

TRY IT

A bag of M&M's contains the following candies: 12 are brown, 20 are yellow, 14 are red, 8 are green, and 16 are orange. If one candy is randomly selected, what is the probability that it's either brown or green?

Overlapping Events

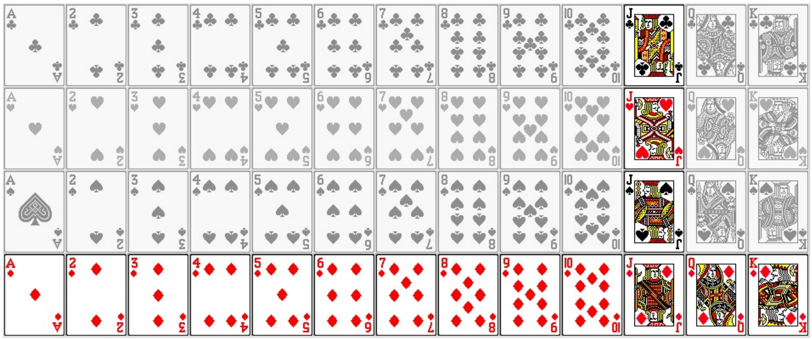
What if the events of interest are not mutually exclusive? How do we compute probabilities of events that are not disjoint? Pictorially, we can visualize this situation with the following diagram, where the red intersection of two circles represents all outcomes when two events both happen. For example, if we consider FCC students, selecting a female and selecting a full-time students would not be mutually exclusive events, since there are certainly female students who go to school full time.



What happens if we try to approach a question about being in A or B by adding the individual probabilities? Notice that if we add the red and blue circles, the purple area in the middle gets counted **twice**, once when we account for the red circle and once for the blue circle. Since this will always happen, we can adjust our answer by simply *subtracting* the overlapping region once from the sum.

As long as we can calculate the overlapping probability (the probably of *both* occurring), this problem isn't much more complicated.

Let's go back to the deck of cards to see an example of how to calculate probabilities with overlapping events. We'll again use the example of drawing a Jack or a diamond.



As we noted already, these are not mutually exclusive events. Because of that, adding the probability of drawing a Jack (4/52) and the probability of drawing a diamond (13/52) gave an incorrect answer of 17/52, where the correct probability–as we noted earlier–is 16/52. Again, this is because we *double counted* the Jack of diamonds, once when we calculated the probability of drawing a Jack and once when we calculated the probability of a diamond.

The way to correct for this double counting is to subtract off the overlap; thus, we'll add up the probability of drawing a Jack and the probability of drawing a diamond, and then subtract the probability of drawing both together (i.e. of drawing the Jack of diamonds):

$$P(J\ OR\ D) = P(J) + P(D) - P(JD) = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52}$$

In general, to calculate probabilities of compound events that are not mutually exclusive, we will use the General Addition rule:

General Addition rule

If A_1 and A_2 are any events, then

$$P(A_1 \text{ or } A_2) = P(A_1) + P(A_2) - P(A_1 \text{ and } A_2)$$

Notice that this is a more general form of the addition rule we stated earlier, with mutually exclusive events. If two events are mutually exclusive, the probability of them occurring together is 0, so the general addition rule simplifies down in that case to the simpler addition rule.

DRAWING A CARD — EXAMPLE 4

Suppose you draw one card from a standard 52-card deck. What is the probability that you get a King or a spade?

Solution

There are 4 Kings and 13 spades, where one of these cards is a King of spades. Drawing the King of spades means both events happen at the same time, so these events are not mutually exclusive. To compute the probability correctly, we need to make sure we don't "double count" any of the outcomes, and in this case it is drawing the King of spades. Applying the general addition rule, we get

$$\begin{aligned} P(K \text{ or } S) &= P(K) + P(S) - P(K \text{ and } S) \\ &= \frac{4}{52} + \frac{13}{52} - \frac{1}{52} \\ &= \boxed{\frac{16}{52} = \frac{4}{13} \approx 0.308} \end{aligned}$$

By subtracting $P(K \text{ and } S)$, we guarantee that we count the King of Spades only once.

TRY IT

Suppose you draw one card from a standard 52-card deck. What is the probability that you get a Queen or a face card?

EXAMPLE 5 FCC STUDENTS

Consider the following information about a group of 130 FCC students:

Gender	Right-handed	Left-handed	Total
Female	58	13	71
Male	47	12	59
Total	105	25	130

If one person is randomly selected from the group, what is the probability this student is female or left-handed?

Solution

These events are not disjoint, since there are 13 females who are left-handed. Thus, we apply the general addition formula:

$$P(F \text{ or } L) = P(F) + P(L) - P(F \text{ and } L)$$

Notice that $P(F)$ and $P(L)$ can be found by looking for the total number of each (the total of the female row and the left-handed column), and $P(F \text{ and } L)$ uses the number of students in the intersection of this row and column.

$$\begin{aligned} P(F \text{ or } L) &= P(F) + P(L) - P(F \text{ and } L) \\ &= \frac{71}{130} + \frac{25}{130} - \frac{13}{130} \\ &= \boxed{\frac{83}{130} \approx 0.638} \end{aligned}$$

Notice that the only students not "qualifying" for the event of interest are right-handed males. There are 47 of them, and 130 - 47 = 83.

We could, of course, solve this problem by highlighting the values in the table that apply, and then simply add them individually:

Gender	Right-handed	Left-handed	Total
Female	[58]	[13]	71
Male	47	[12]	59
Total	105	25	130

Adding these up gives the same total for the number of qualifying students.

$$58 + 13 + 12 = 83$$

Once again, there are often many ways to solve a probability problem; our goal is to have as many tools as possible, so that we can select the easiest approach.

TRY IT

Using the example above, compute the probability of selecting a male or a right-handed student.

SPEEDING TICKETS AND CAR COLOR

EXAMPLE 6

The table below shows the number of survey subjects who have received a speeding ticket in the last year and those who have not received a speeding ticket, as well as the color of their car. Find the probability that a randomly chosen person has a red car *or* got a speeding ticket

	Speeding Ticket	No Speeding Ticket	Total
Red Car	15	135	150
Not Red Car	45	470	515
Total	60	605	665

Solution

Notice that having a red car and getting a speeding ticket are not mutually exclusive events, since 15 people had both. Thus, we perform the following computations:

$P(red\ car) + P(got\ a\ speeding\ ticket) - P(red\ car\ and\ got\ a\ speeding\ ticket)$

Once again, we simply need to add the totals for the red car row and the speeding ticket column, then subtract the value in the intersection of these (and then of course divide this by the total number of observations).

$$\frac{150}{665} + \frac{60}{665} - \frac{15}{665} = \boxed{\frac{195}{665} \approx 0.293}$$

TRY IT

A local fitness club conducted a survey about the type of workouts their members prefer, and the results are recorded in the table below.

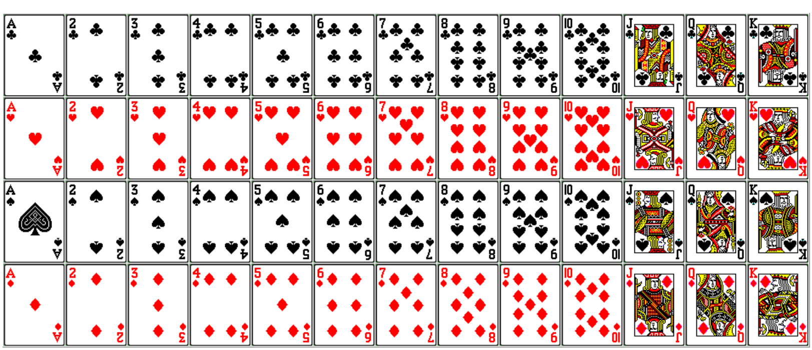
	Cardio Exercises	Strength Training	Total
Female	45	16	61
Male	27	82	109
Total	72	98	170

If one person is randomly selected from this group, what is the probability of selecting a male or a member who prefers cardio exercise?

Complements

The probability of an event not occurring can be just as useful as computing the probability of that event happening. The best way to introduce this concept is to consider an example. Let's revisit the standard 52-card deck, where we randomly select one card:

Remember, there are generally multiple ways to solve a probability problem. Here we'll find a valuable tool that allows us to look at problems from a new angle.



What is the probability of not drawing an Ace? Well, you know that there are 4 Aces in the deck, so $52 - 4 = 48$ cards that are not Aces. We compute:

$$P(\text{not Ace}) = \frac{48}{52} \approx 0.923$$

Now, notice that

$$\frac{48}{52} = 1 - \frac{4}{52}, \text{ where } P(Ace) = \frac{4}{52}$$

This is not a coincidence. If you recall the basic rules of probability, the sum of the probabilities of all outcomes must be 1. In this case, the card you draw is either Ace or it's not, so it makes sense that the probabilities of these two events add up to 1.

Complement of an event

The complement of an event A is denoted by A^c and represents all outcomes not in A.

1. $P(A) + P(A^c) = 1$
2. $P(A) = 1 - P(A^c)$
3. $P(A^c) = 1 - P(A)$

EXAMPLE 7 NOT HEARTS!

If you pull a random card, what is the probability it is not a heart?

Solution

There are 13 hearts in the deck, so $P(\text{hearts}) = 13/52 = 1/4$. The probability of not drawing a heart is the complement:

$$\begin{aligned} P(\text{not hearts}) &= 1 - P(\text{hearts}) \\ &= 1 - \frac{1}{4} \\ &= \boxed{\frac{3}{4} = 0.75} \end{aligned}$$

TRY IT

If you pull a random card, what is the probability it is not a face card?

Let's consider an example that might be more relevant to you as a student:

EXAMPLE 8 MULTIPLE CHOICE QUESTION

A multiple choice question has 5 answers, and exactly one of them is correct. If you were to guess, what is the probability of not getting the correct answer?

Solution

Since only one of the answers is correct, we have $P(\text{correct}) = 1/5$, so

$$\begin{aligned} P(\text{not correct}) &= 1 - P(\text{correct}) \\ &= 1 - \frac{1}{5} \\ &= \boxed{\frac{4}{5} = 0.8} \end{aligned}$$

TRY IT

A multiple choice question has 6 answers, and exactly two of them are correct. If you were to guess, what is the probability of not getting the correct answer? If a test has 4 questions with 6 possible answers, what is the probability of not getting any of the questions correct? (*You are technically not ready to answer this part of the question just yet, but it is a good exercise to think about!*)

FCC STUDENT DEMOGRAPHICS

EXAMPLE 9

According to the FCC website, female students made up 57% of the Fall 2014 student body. If one student is randomly selected, what is the probability the student is not female?

Solution

The probability of selecting a female student is 0.57, so using the complement rule, we compute:

$$
\begin{aligned}
P(\text{not female}) &= 1 - P(\text{female}) \\
&= 1 - 0.57 = \boxed{0.43}
\end{aligned}
$$

TRY IT

According to the FCC website, full-time students made up 34% of the fall 2014 student body. If one student is randomly selected, what is the probability the student is not full-time?

Exercises 4.2

1. A jar contains 6 red marbles numbered 1 to 6 and 8 blue marbles numbered 1 to 8. A marble is drawn at random from the jar. Find the probability the marble is red or odd-numbered.

2. A jar contains 4 red marbles numbered 1 to 4 and 10 blue marbles numbered 1 to 10. A marble is drawn at random from the jar. Find the probability the marble is blue or even-numbered.

3. You draw one card from a standard 52-card deck.

(a) What is the probability of selecting a King or a Queen?

(b) What is the probability of selecting a face card or a 10?

(c) What is the probability of selecting a spade or a heart?

(d) What is the probability of selecting a red card or a black card?

4. You are dealt a single card from a standard 52-card deck.

(a) Find the probability that you are not dealt a diamond.

(b) Find the probability that you are not dealt a face card.

(c) Find the probability that you are not dealt an Ace.

(d) Find the probability that you are not dealt a jack or a king.

5. Consider the following information about a group of 130 FCC students:

Gender	Right-handed	Left-handed	Total
Female	58	13	71
Male	47	12	59
Total	105	25	130

If one person is randomly selected from the group, what is the probability this student is female or left-handed?

6. The table below shows the number of survey subjects who have received and not received a speeding ticket in the last year, and the color of their car. Find the probability that a randomly chosen person has a red car *or* got a speeding ticket.

	Speeding Ticket	No Speeding Ticket	Total
Red Car	15	135	150
Not Red Car	45	470	515
Total	60	605	665

7. Suppose you roll a blue six sided die and a red six sided die, and add their totals. Find the probability of rolling:

(a) A 7 or 11.

(b) An even number or a number less than 6.

(c) A prime number or a number greater than 5.

8. A bag contains 4 white counters, 6 black counters, and 1 green counter. What is the probability of drawing:

(a) A white counter or a green counter?

(b) A black counter or a green counter?

(c) Not a green counter?

9. A poll was taken of 14,056 working adults aged 40-70 to determine their level of education. The participants were classified by sex and by level of education. The results were as follows.

Education Level	Male	Female	Total
High School or Less	3141	2434	5575
Bachelor's Degree	3619	3761	7380
Master's Degree	534	472	1006
Ph.D.	52	43	95
Total	7346	6710	14,056

A person is selected at random. Compute the following probabilities:

(a) The probability that the selected person does not have a Ph.D.

(b) The probability that the selected person does not have a Master's degree.

(c) The probability that the selected person is female or has a Master's degree.

(d) The probability that the selected person is male or has a Ph.D.

SECTION 4.3 The Multiplication Rule and Conditional Probability

We began by calculating the probabilities of single events occurring, and then we learned how to combine events using *OR*. Now we ask a different question: suppose we know how to calculate the probability of A and the probability of B on their own; how can we calculate the probability that A *AND* B both occur? To set this up, we'll look at two situations: flipping a coin twice and drawing two cards *without replacement* (this will be important).

Flipping a coin twice If we flip a coin twice in succession, the sample space is

$$S = \{HH, HT, TH, TT\}.$$

Now suppose we ask the following questions:

Independent events: don't affect each other (whether the first flip is heads or tails, the probabilities for the second flip are not impacted)

1. What is the probability that the first flip results in a head?

   Either by noticing that there are two possibilities for the first flip or by looking at the sample space and seeing that there are two outcomes (out of four total) that correspond to a head on the first flip, we can reason that this probability is 1/2.

2. What is the probability that the second flip results in a tail?

   Using the same reasoning, we conclude that this probability is also 1/2.

3. What is the probability that the first flip results in a head *AND* the second flip results in a tail?

   Looking at the sample space, we notice that there is exactly one outcome that corresponds to this (out of four), so this probability is 1/4.

Notice that the probability of both happening together is the probability of one times the probability of the other:

$$\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

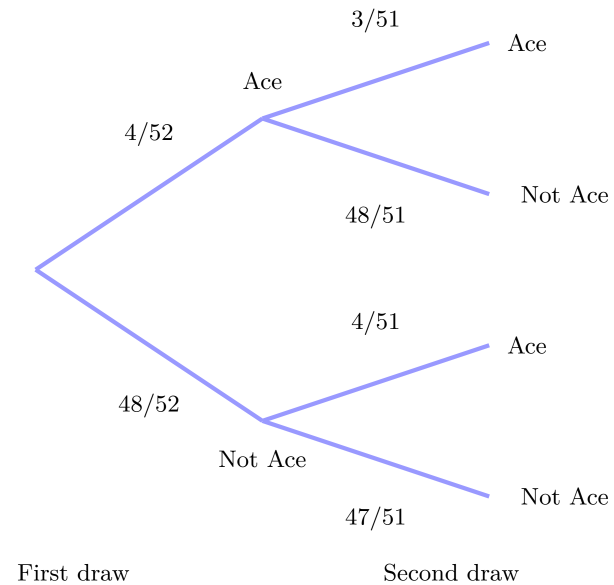
Seeing this, and noting the title of the section, we may be tempted to jump to the conclusion that the probability of A *AND* B is simply the probability of A times the probability of B. However, the next scenario illustrates that we need to be a bit more careful.

Just as we found with the addition rule, there is a simple version that works if a certain condition is met, and if not, there is a more general version of the multiplication rule.

Drawing two cards without replacement Suppose we draw one card, and then *without* placing it back and re-shuffling the deck, we draw a second card. What is the probability that we draw two Aces?

Dependent events: affect each other (after pulling out the first card, the deck has changed, so the probabilities have shifted)

This situation is different from the previous one, because now what happens on the first draw affects the probabilities for the second draw. In other words, the probability of drawing an Ace the first time is 4/52. If we draw an Ace the first time, there are only 3 Aces left and 51 total cards left, so the probability of drawing an Ace the second time is 3/51. However, if we do not draw an Ace the first time, there are still 4 Aces in the deck, so the probability of drawing an Ace the second time is 4/51. We can illustrate this with a branching tree diagram.



Now the probability of drawing an Ace both times is the probability of drawing an Ace the first time multiplied by the probability of drawing an Ace the second time **given that we drew an Ace the first time**. Notice on the tree diagram that this corresponds to following the upward branch both times.

This is because *only* if we draw an Ace the first time do we have any chance of fulfilling the scenario; if we fail to draw an Ace the first time, it doesn't matter what we do the second time–we've already failed.

Thus, the probability of drawing an Ace both times is

$$P(\text{Ace the first time}) \cdot P(\text{Ace the second time IF we drew one the first time})$$
$$= \frac{4}{52} \cdot \frac{3}{51} = \frac{12}{2652} \approx 0.0045$$

This is what we call *conditional probability*, and it's what we have to consider for the general multiplication rule.

Independence

What was the difference between those two scenarios? Why, in the first one, could we simply multiply the individual probabilities and in the second we had to think about conditional probability? The answer lies in what we call **independence**: when flipping the coin, each time we flipped it had no impact on the other times; when we drew the cards without replacement, though, one draw affected the next. Notice that we made the careful distinction that we drew without replacement; if we had replaced the first card and re-shuffled the deck before drawing again, the two draws would have been independent.

Independence

Two events are independent if the outcome of one has no effect on the probability of the other occurring.

Note that saying that two events are *independent* is different than saying that two events are *mutually exclusive*.

- If two events are independent, they have no effect on each other's likelihood of occurring.
- If two events are mutually exclusive, they cannot occur together, so they do have an effect on each other's likelihood of occurring (namely, making it impossible).

INDEPENDENT EVENTS — EXAMPLE 1

Determine whether these events are independent:

(a) A fair coin is tossed two times. The two events are A = first toss is Heads and B = second toss is Heads.

Solution The probability that Heads comes up on the second toss is 1/2 regardless of whether or not Heads came up on the first toss, so these events are independent.

(b) The two events A = *It will rain tomorrow in Frederick, MD* and B = *It will rain tomorrow in Thurmont, MD*

Solution These events are not independent because it is more likely that it will rain in Thurmont on days it rains in Frederick.

(c) You draw a red card from a deck, then draw a second card without replacing the first.

Solution These events are dependent, specifically because the first card is *not replaced*. After drawing the first card, the deck looks different than it did before. There are 25 red cards left and 26 black cards left, so the probabilities have shifted.

(d) You draw a face card from the deck, then replace it and re-shuffle the deck before drawing a second card.

Solution Since you reset the deck between draws, the events are independent.

Now we are ready to formally state the rule that we used in the first scenario at the beginning of the section.

The Multiplication Rule for Independent Events

Probabilities of independent events

If A and B are independent, then the probability of both A and B occurring is

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

We can generalize this to finitely many independent events $A_1, A_2, \ldots, A_k$

$$P(A_1 \text{ and } A_2 \text{ and } \ldots \text{ and } A_k) = P(A_1) \cdot P(A_2) \cdot \ldots \cdot P(A_k)$$

COINS AND DICE — EXAMPLE 2

Suppose you flip a coin and roll a six-sided die once. What is the probability you get Tails and an even number?

Solution Flipping a coin and rolling a die are independent events, since the outcome of one does not effect the outcome of the other. Thus, we compute it as follows:

$$\begin{aligned} P(T \text{ and } \textit{even number}) &= P(T) \cdot P(\textit{even number}) \\ &= \frac{1}{2} \cdot \frac{3}{6} \\ &= \frac{3}{12} = \boxed{\frac{1}{4} = 0.25} \end{aligned}$$

TRY IT

Assume you have a 52 card deck, and you select two cards at random. Also assume that you replace and reshuffle after each selection. Find the probability of drawing a king first and then a black card.

EXAMPLE 3

LEFT-HANDED POPULATION

Unlike a deck of cards, in which removing one card makes a noticeable difference, removing one person from hundreds of millions makes such a small difference that we can ignore it for simplicity.

About 9% of people are left-handed. Suppose 2 people are selected at random from the U.S. population. Because the sample size of 2 is very small relative to the population, it is reasonable to assume these two people are independent. What is the probability that both are left-handed?

The probability the first person is left-handed is 0.09, which is the same for the second person:

$$P(\text{both left}) = 0.09 \cdot 0.09 = \boxed{0.0081}$$

TRY IT

According to the US Census, in 2009 86.1% of working adults commuted in a car, truck, or van. If three people are selected from the population of working adults, what is the probability that all three commuted in a car, truck, or van?

EXAMPLE 4

BOYS AND GIRLS

Assuming that probability of having a boy is 0.5, find the probability of a family that has 3 children having 3 boys.

Solution

Since the gender of each child is independent, we use the multiplication formula for independent events:

$$\begin{aligned} P(3\ \mathit{boys}) &= P(\mathit{boy}) \cdot P(\mathit{boy}) \cdot P(\mathit{boy}) \\ &= 0.5 \cdot 0.5 \cdot 0.5 = \boxed{0.125} \end{aligned}$$

TRY IT

In your drawer you have 10 pairs of socks, 6 of which are white, and 7 tee shirts, 3 of which are white. If you randomly reach in and pull out a pair of socks and a tee shirt, what is the probability that both are white?

The Multiplication Rule for Dependent Events

In the second scenario at the beginning of the section, where the events were not independent, we found that we could calculate the probability of both happening by multiplying the probability of the first by the probability that the second occurred IF the first had happened. We call this **conditional probability**: the probability that B happens on the *condition* that A already happened.

The notation we use is

$$P(B|A).$$

For example, in the scenario where we wanted to draw two Aces in a row, we could write the conditional probability for the second draw as

$$P(\text{ace on second draw} \mid \text{ace on first draw})$$

The vertical bar | is read as "given," so the above expression is short for "The probability that an ace is drawn on the second draw given that an ace was drawn on the first draw."

As we noted earlier, after an ace is drawn on the first draw, there are 3 aces out of 51 total cards left. This means that the conditional probability of drawing an ace after one ace has already been drawn is $3/51 = 1/17$. Thus, the probability of both cards being aces is

$$\frac{4}{52} \cdot \frac{3}{51} = \frac{12}{2652} = \frac{1}{221}$$

Multiplication formula for dependent events

If events A and B are not independent, then

$$P(A \text{ and } B) = P(A) \cdot P(B|A)$$

Note that this, like with the addition rule, is the general multiplication rule; if A and B are independent, $P(B|A) = P(B)$ (because the probability of B is the same regardless of whether A has occurred or not) and the general multiplication formula becomes the simpler form for independent events that we have already seen.

DRAWING CARDS WITHOUT REPLACEMENT — EXAMPLE 5

If you pull 2 cards out of a deck, what is the probability that both are spades?

Solution

The probability that the first card is a spade is 13/52, while the probability that the second card is a spade, given the first was a spade, is 12/51. Thus, the probability that both cards are spades is

$$P(2\ spades) = \frac{13}{52} \cdot \frac{12}{51}$$
$$= \boxed{\frac{156}{2652} \approx 0.0588}$$

TRY IT

In your drawer you have 10 pairs of socks, 6 of which are white. If you reach in and randomly grab two pairs of socks, what is the probability that both are white?

M&M'S — EXAMPLE 6

A bag of M&M's contains the following breakdown of colors:

Red	Yellow	Brown	Blue	Orange	Green
12	18	24	22	13	17

Suppose you pull two M&M's out of the bag (without replacing the candy after each pull). Find the following probabilities:

(a) The probability of drawing two red candies

(b) The probability of drawing a blue candy and then a brown candy, in that order

(c) The probability of **not** drawing two green candies

(d) The probability of drawing one orange candy and one yellow candy (note that order is not mentioned)

Solution

(a) Since there are a total of 106 candies at first, the probability of drawing a red candy on the first try is 12/106. Assuming that first try is successful, there will be a total of 105 candies left, of which 11 are red, so the probability of drawing a red candy on the second try will be 11/105:

$$\begin{aligned} P(red,\ red) &= P(red) \cdot P(red \mid red) \\ &= \frac{12}{106} \cdot \frac{11}{105} \\ &= \boxed{\frac{132}{11,130} \approx 0.0119} \end{aligned}$$

(b) At first, there will be 22 blue candies out of a total of 106, and if the first try yields a blue one, all the brown ones (24) will still be in the bag, which will then contain 105 in total:

$$\begin{aligned} P(blue,\ brown) &= P(blue) \cdot P(brown \mid blue) \\ &= \frac{22}{106} \cdot \frac{24}{105} \\ &= \boxed{\frac{528}{11,130} \approx 0.0474} \end{aligned}$$

(c) In order to *not* draw two green candies, we could start thinking of all the possible combinations *other* than *green, green*, but it's much easier to use the complement rule. We can calculate the probability of drawing two green candies the same way we did at the beginning with red candies, then subtract this answer from 1:

$$\begin{aligned} P(not\ 2\ green) &= 1 - P(2\ green) \\ &= 1 - P(green) \cdot P(green \mid green) \\ &= 1 - \frac{17}{106} \cdot \frac{16}{105} \\ &= 1 - \frac{272}{11,130} \\ &= \boxed{\frac{10,858}{11,130} \approx 0.9756} \end{aligned}$$

This is important: it is much easier to calculate the probability of drawing two green candies first, and then subtracting this from one. If we didn't do this, we would have to calculate three separate probabilities and add them together:

- Drawing a green, then a non-green candy — $(17/106) \cdot (89/105)$
- Drawing a non-green, then a green candy — $(89/106) \cdot (17/105)$
- Drawing a non-green, then a non-green candy — $(89/106) \cdot (88/105)$

(d) This time, order is not significant, so there are two possibilities:

- Draw orange, then yellow
- Draw yellow, then orange

Since these two outcomes are mutually exclusive, the final answer will be the sum of these two. To calculate each individual answer, we can use the same approach we used with the blue and brown example:

$$\begin{aligned} P(orange\ and\ yellow) &= P(orange,\ yellow) + P(yellow,\ orange) \\ &= P(orange) \cdot P(yellow \mid orange) + P(yellow) \cdot P(orange \mid yellow) \\ &= \frac{13}{106} \cdot \frac{18}{105} + \frac{18}{106} \cdot \frac{13}{105} \\ &= \frac{234}{11,130} + \frac{234}{11,130} \\ &= \boxed{\frac{468}{11,130} \approx 0.0420} \end{aligned}$$

Notice that the answer for each part was the same, so we could also have simply calculated the probability of one ordering, then doubled the answer.

It is often useful to be able to calculate conditional probabilities using contingency tables, so let's take a look at an example of that.

The key here is that when we encounter the word *given*, it means that we will restrict ourselves only to that category, meaning that the denominator will not be the total number of observations, but rather the total for the relevant row or column.

CONDITIONAL PROBABILITY AND CONTINGENCY TABLES — EXAMPLE 7

We will again use the data regarding 130 FCC students, broken down by gender and dominant hand:

Gender	Right-handed	Left-handed	Total
Female	58	13	71
Male	47	12	59
Total	105	25	130

(a) What is the probability that a randomly chosen student is female, given that the student is left-handed?

(b) What is the probability that a randomly chosen student is right-handed, given that the student is male?

Solution

1. To calculate conditional probabilities from a contingency table, all we have to do is restrict ourselves to the "given" category. For this one, we are given that the student is left-handed, so we'll only look at the left-handed column and see what proportion of those are female. The total number of left-handed students is 25, and within that group, 13 are female and 12 are male:

$$P(female \mid left) = \boxed{\frac{13}{25} = 0.52}$$

2. Here we'll only look at the male row, since we're given that the randomly chosen student is male. All we need to calculate is what proportion of males in this group are right-handed:

$$P(right \mid male) = \boxed{\frac{47}{59} \approx 0.7966}$$

We'll wrap up this section with a short discussion of a surprising application of conditional probability: medical tests.

Let's say, for instance, that a certain disease infects 100 out of 100,000 people. There is a test for this disease, and it is correct 99% of the time. Now, here's the question: if you go to the doctor and take this test, and the result is positive (meaning the test says that yes, you have the disease), how likely is it that you truly have it?

The intuitive answer is that it must be 99% likely that you have the disease, since the test is correct 99% of the time. However, the truth is more complicated.

The key to this problem is that the disease is fairly rare. So when a test comes back positive, there are two possibilities: either you have the disease or the test is wrong. Notice that there is only a 1 in 1000 chance that you have the disease, while there is a 1 in 100 chance that the test is wrong. Therefore, it turns out that *even with a positive result, you are more likely to be healthy.* Of course, this ignores the fact that you may have only gotten the test because you were showing symptoms, in which case the odds of having the disease would be much higher than 1 in 1000.

To get a more precise answer, we can build a contingency table like the one in the last example. Let's start with a group of 100,000 people: 100 of them are sick and 99,900 are healthy.

	Sick	Healthy	Total
Positive Test			
Negative Test			
Total	100	99,900	100,000

Now, let's imagine giving everyone in this group a test (remember, the test is correct 99% of the time and wrong 1% of the time). In the sick group, how many would test positive and how many would test negative? Since the correct result for this group is a positive test, 99 of them would get a positive result and 1 would get a negative result.

	Sick	**Healthy**	**Total**
Positive Test	99		
Negative Test	1		
Total	100	99,900	100,000

What about the healthy group? For them, a positive test is the wrong result, so 1% of them (999 people) would see that, while the remainder would get a correct negative result.

	Sick	**Healthy**	**Total**
Positive Test	99	999	1098
Negative Test	1	98,901	98,902
Total	100	99,900	100,000

Now that we've filled out all the possibilities, we can return to the question: suppose you get a positive result; what's the probability that you're sick? We can state this more precisely using conditional probability: what is the probability that you are sick, *given* that you have a positive test?

Using the approach of the last example, we can do this by focusing on the row that shows everyone with a positive test, and out of that group, find the probability of being sick:

$$P(sick \mid positive\ result) = \frac{99}{1098} \approx 0.09$$

There's only about a 9% chance that you are sick, *even after getting a positive test.* Again, the reason for this is that this disease is *known* to be rare, while an incorrect test result is not as rare.

Exercises 4.3

1. You have a box of chocolates that contains 50 pieces, of which 30 are solid chocolate, 15 are filled with cashews and 5 are filled with cherries. All the candies look exactly alike. You select a piece, eat it, select a second piece, eat it, and finally eat one last piece. Find the probability of selecting a solid chocolate followed by two cherry-filled chocolates.

2. You roll a fair six-sided die twice. Find the probability of rolling a 6 the first time and a number greater than 2 the second time.

3. A math class consists of 25 students, 14 female and 11 male. Three students are selected at random, one at a time, to participate in a probability experiment. Compute the probability that:

(a) A male is selected, then two females.

(b) A female is selected, then two males.

(c) Two females are selected, then one male.

(d) Three males are selected.

(e) Three females are selected.

4. A large cooler contains the following drinks: 6 lemonade, 8 Sprite, 15 Coke, and 7 root beer. You randomly pick two cans, one at a time (without replacement).

(a) What is the probability that you get 2 cans of Sprite?

(b) What is the probability that you do not get 2 cans of Coke?

(c) What is the probability that you get either 2 root beer or 2 lemonade?

(d) What is the probability that you get one can of Coke and one can of Sprite?

(e) What is the probability that you get two drinks of the same type?

5. My top drawer contains different colored socks: 14 are white, 10 are black, 6 are pink, and 4 are blue. All socks in the drawer are loose. Every morning I randomly select 2 socks, one at a time. Calculate the following probabilities, giving both fraction and decimal answers, rounding to 4 decimal places:

(a) What is the probability that I get a blue pair of socks?

(b) What is the probability that I do not get a blue pair of socks?

(c) What is the probability that I either get a white pair or a blue pair of socks?

(d) What is the probability that I get one black sock and one white sock?

6. Suppose a math class contains 25 students, 14 females (three of whom speak French) and 11 males (two of whom speak French). Compute the probability that a randomly selected student speaks French, given that the student is female.

7. Suppose a math class contains 30 students, 18 females (four of whom speak French) and 12 males (three of whom speak French). Compute the probability that a randomly selected student is male, given that the student speaks French.

8. A certain virus infects one in every 400 people. A test used to detect the virus in a person is positive 90% of the time if the person has the virus and 10% of the time if the person does not have the virus.

(a) Find the probability that a person has the virus given that they have tested positive.

(b) Find the probability that a person does not have the virus given that they test negative.

9. A poll was taken of 14,056 working adults aged 40-70 to determine their level of education. The participants were classified by sex and by level of education. The results were as follows.

Education Level	Male	Female	Total
High School or Less	3141	2434	5575
Bachelor's Degree	3619	3761	7380
Master's Degree	534	472	1006
Ph.D.	52	43	95
Total	7346	6710	14,056

A person is selected at random. Compute the following probabilities:

1. The probability that the selected person is male, given he has a Master's degree.
2. The probability that the selected person does not have a Master's degree, given it is a male.
3. The probability that the selected person is female, given that she has a Bachelor's degree.
4. The probability that the selected person has a Ph.D, given it is a female.

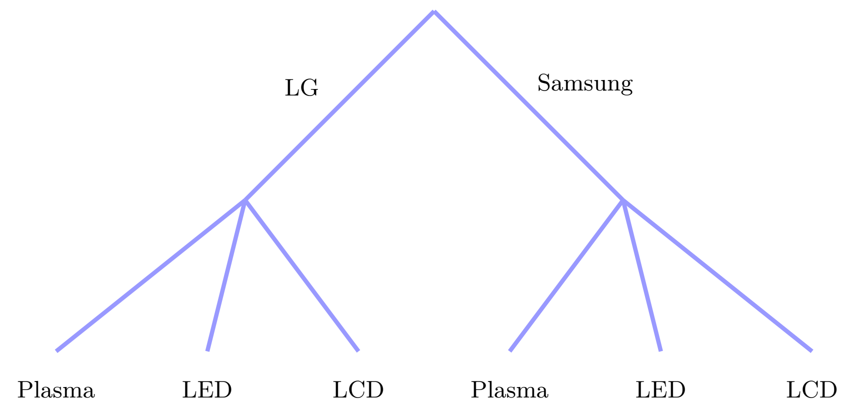
SECTION 4.4 Counting Methods

Counting? You already know how to count or you wouldn't be taking a college-level math class, right? Well yes, but what we'll really be investigating here are ways of counting efficiently. When we get to the probability situations a bit later in this chapter we will need to count some very large numbers, like the number of possible winning lottery tickets. One way to do this would be to write down every possible set of numbers that might show up on a lottery ticket, but believe me: you don't want to do this.

In this section we will see how to count the number of ways that something could happen without listing them all out. This is because when we calculate probabilities we really just need to count the number of possible outcomes in a sample space and the number of outcomes that correspond to an event that we're interested in.

Fundamental Counting Principle

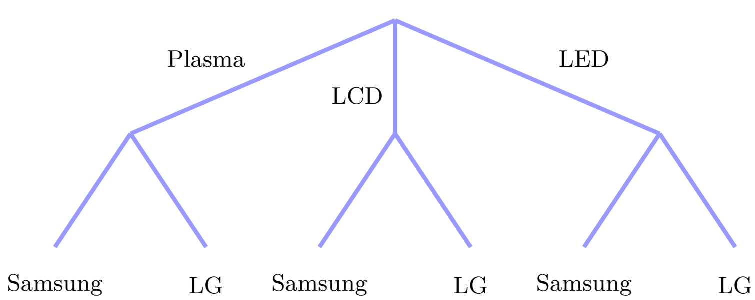
Suppose you went to buy a new TV, and you narrowed your choices down to two brands: LG and Samsung. You already have the size picked out, but within each of those brands, you need to choose among plasma, LED, and LCD. How many total choices do you have?



By counting the number of branches that our decision could follow, it's clear that there are six total possibilities: for each choice of brand, there are three possibilities, so there are

$$2 \times 3 = 6 \text{ choices.}$$

Notice that if we switch the order of the decisions, we get the same number of final options:



We can generalize this to get the **fundamental counting principle.**

Fundamental Counting Principle

If we are asked to choose one item from each of two separate categories where there are m items in the first category and n items in the second category, then the total number of available choices is

$$m \times n$$

We can generalize this principle to finitely many categories.

For instance, if we want to count the number of possible lottery tickets that could be made with numbers that have 4 digits, all we have to do is multiply together the possible values for each digit (note that there are 10 possibilities: 0–9).

$$\underline{10} \cdot \underline{10} \cdot \underline{10} \cdot \underline{10} = 10,000 \text{ possible lottery tickets}$$

We are beginning to see the value of such a simple principle for counting; we didn't have to list out all 10,000 possibilities, but we were able to make a quick calculation and know that that's how many there are.

EXAMPLE 1 SO MUCH READING!

There are 21 novels and 18 volumes of poetry on a reading list for a college English course. How many different ways can a student select one novel and one volume of poetry to read during the quarter?

Solution Since there are 21 choices for which novel to pick and 18 choices for which poetry volume to pick, there are a total of

$$\underline{21} \cdot \underline{18} = \boxed{378 \text{ choices}}$$

Note that the order in which we make the decision doesn't matter, since $21 \cdot 18 = 18 \cdot 21$.

TRY IT Suppose at a particular restaurant you have three choices for an appetizer (soup, salad or bread-sticks), five choices for a main course (hamburger, sandwich, quiche, fajita or pasta) and two choices for dessert (pie or ice cream). If you are allowed to choose exactly one item from each category for your meal, how many different meal options do you have?

EXAMPLE 2 PIZZA TOPPINGS

Assume you work at a pizza parlor, and you are offering a special on large, two topping pizzas. Your toppings are broken up into two categories:

Choice of meat	Choice of veggies
Pepperoni	Green peppers
Sausage	Tomatoes
Ham	Onions
Grilled chicken	

The toppings must be chosen with one from each category. How many different two-topping pizzas can you make?

Solution Here, there are 4 meat choices and 3 veggie choices, for a total of

$$\underline{4} \cdot \underline{3} = \boxed{12 \text{ choices}}$$

TRY IT Assume that you have expanded the special so that you also receive a 2-liter bottle of soda with your large pizza. Assume your possible drink choices are Pepsi, Diet Pepsi, Mountain Dew and Root Beer. Now how many different dinner specials can you have, including pizza and drinks?

We'll do one last example with the fundamental counting principle, but once again, everything boils down to counting the number of possibilities in each category or each stage of the decision-making process. As long as we can do that, counting the total number of possibilities just involves multiplying those together.

APARTMENT SHOPPING — EXAMPLE 3

An apartment complex offers apartments with four different choices: the number of bedrooms, number of bathrooms, floor, and view.

Bedrooms	Bathrooms	Floor	View
1	1	first	Lake view
2	2	second	Golf view
3			No view

How many apartment options are available?

Solution

Since we are making 4 choices, we'll be multiplying 4 numbers together: the number of options for each choice.

$$\underline{3 \text{ numbers of bedrooms}} \times \underline{2 \text{ numbers of bathrooms}} \times \underline{2 \text{ numbers of floors}} \times \underline{4 \text{ numbers of views}} = \boxed{36 \text{ options}}$$

TRY IT

You know that you have a multiple choice exam coming up, and you figure you don't need to study too hard since it is multiple choice. But then you remember the Fundamental Counting Principle and you decide you better check how many possible ways there are for you to answer the questions. The exam consists of 10 questions, with each question having 4 possible choices and only one correct answer per question. If you select one of these 4 choices for each question and leave nothing blank, how many ways can you answer the questions? How many ways are there to get a perfect exam?

The fundamental counting principle can even handle questions that sound difficult when they're posed. For instance, suppose you find yourself in a group of five friends going to a movie. Two of the five are arguing, and they demand to sit on opposite sides of the row of five seats. How many ways are there to arrange this group? Rather than trying to list all the possibilities, all we have to think through is how many options there are for who can sit in each seat.

1. Only one of the two arguing friends can sit in the first seat, so this seat has two options.
2. One person has sat down, leaving four standing. The other arguing member, though, cannot sit in the second seat, leaving three options for it.
3. Of the three still standing after the first two seats are filled, only two can sit in the third seat.
4. There's therefore only one option for the fourth seat.
5. Finally, the fifth seat only has one option as well: the other warring member.

Thus there are a total of

$$\underline{2} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1} \cdot \underline{1} = 12 \text{ possibilities}$$

Factorials!

In many of the examples that we'll see next, it will be convenient to have notation for what we call **factorials**. To see this, consider the following example: you want to know how many ways there are to arrange your seven textbooks on the shelf in seven positions. Following the pattern of the example with the friends sitting at the movie theater, we can see that there are 7 options for the first position, 6 options for the second position (since one has already been placed), 5 options for the third position, and so on. There are a total of

$$\underline{7} \cdot \underline{6} \cdot \underline{5} \cdot \underline{4} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1} = 5040 \text{ arrangements}$$

Any time we start arranging things in order, we'll see a descending product like this one, so for simplicity's sake, we define that as a factorial (so that we can more easily tell our calculators to calculate it, for instance, and so that our formulas are more concise).

The notation we use is the exclamation mark, so "7 factorial" would be written 7!:

$$7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

Factorials

The **factorial** of any positive integer n is defined as the product of every integer from n down to 1:

$$n! = n \cdot (n-1) \cdot (n-2) \cdots (3) \cdot (2) \cdot (1)$$

By definition, $0! = 1$.

EXAMPLE 4 MOVIE MARATHON

You have been given the job of scheduling the movies for the FCC Movie Marathon. You have 4 choices for movies: one action movie, one comedy, one drama, and one horror. Luckily for you, you know the Fundamental Counting Principle. How many different ways are there to order these 4 movies?

Solution There are 4 ways to pick the first movie, then 3 ways to pick the second, 2 ways to pick the third, and only 1 way to pick the fourth. Now we have factorial notation for this:

$$4! = 4 \cdot 3 \cdot 2 \cdot 1 = \boxed{\text{24 possibilities}}$$

TRY IT

Going back to the movie marathon, since you know that most people don't like Horror movies you decide that it should go last. With this in mind, how many different ways is there to arrange the movie marathon?

Permutations

Permutations arise whenever we want to arrange items in order, like in the example of arranging seven textbooks on a shelf. Not only that, but we can also calculate the number of ways to select, for instance, seven textbooks out of a pool of ten and *then* arrange them in order.

Let's think about that example: since we're arranging seven books, there are seven slots to fill. In the first slot, we have 10 options (the full list of books available), then 9 in the next slot, and so on:

$$\underline{10} \cdot \underline{9} \cdot \underline{8} \cdot \underline{7} \cdot \underline{6} \cdot \underline{5} \cdot \underline{4}$$

Notice how this looks like a factorial calculation, 10!, but it has been cut off early. We could always calculate permutations using this same approach, but that cut-off factorial gives us an idea for a shortcut: notice that if we started with 10!, then divided by 3!, that would cut off the tail of the factorial after the 4:

$$\frac{10!}{3!} = \frac{\underline{10} \cdot \underline{9} \cdot \underline{8} \cdot \underline{7} \cdot \underline{6} \cdot \underline{5} \cdot \underline{4} \cdot \cancel{\underline{3} \cdot \underline{2} \cdot \underline{1}}}{\cancel{\underline{3} \cdot \underline{2} \cdot \underline{1}}} = \underline{10} \cdot \underline{9} \cdot \underline{8} \cdot \underline{7} \cdot \underline{6} \cdot \underline{5} \cdot \underline{4}$$

Essentially, to find the number of ways to choose 7 and arrange them, we divide the number of ways to arrange the 10 (10!) by the number of ways to arrange the 3 leftovers (3!). This is how we calculate permutations, so the number of ways to select 7 textbooks out of 10 and arrange them is

$$\frac{10!}{3!}$$

Notice that the 10 is the size of the pool that we can choose from, and the 3 is the number of leftovers, the difference between the total number of available options and the number that we chose.

Permutations

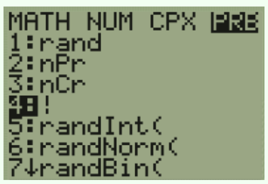
We say that there are nPr permutations of size r that may be selected from among n choices without replacement when **order matters**. To compute the number of permutations of r items from a collection of n items, we use the formula below:

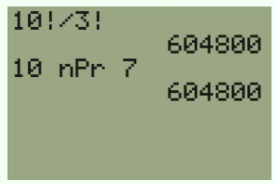
$$nPr = \frac{n!}{(n-r)!}$$

To simplify the process, we can use a calculator to solve problems like this; many have the permutation formula built in.

Using Your Calculator

There are two ways to use your calculator to solve problems involving permutations (and later, combinations): enter the formula with the factorials, or use the built-in permutation function. Both are found by pressing the MATH button and scrolling over to the `PRB` (probability) menu.





Note that to use the **`nPr`** option, type in the value of n first, then select **`nPr`**, then enter the value of r and press ENTER.

ARRANGING CD'S — EXAMPLE 5

You have 18 CDs, and you need to arrange 8 of your favorites on the shelf near your stereo. How many ways can you select and arrange the CDs, assuming that the order of the CDs makes a difference to you?

Solution

Using the formula for permutations, n here is 18 (the number that we can choose from) and r is 8 (the number that we are selecting to organize):

$$\begin{aligned} nPr &= \frac{18!}{(18-8)!} \\ &= \frac{18!}{10!} \\ &= \boxed{1,764,322,560 \text{ possibilities}} \end{aligned}$$

TRY IT

How many arrangements can be made using four of the letters of the statement MATH RULES if no letter is to be used more than once and the space is not considered? (For this we don't care if it actually makes a word, so "AUET" would be one of those 4 letter permutations.)

In case you weren't convinced before, hopefully you agree now that these counting principles are useful, because if you had to list all 1,764,322,560 options in order to count them, you could list one every second and still spend almost 56 years doing so. Instead, we were able to apply a counting principle and get that result by having our calculator do the arithmetic for us.

EXAMPLE 6 **FCC MATH CLUB**

The math club has 18 members. According to the bylaws they need to have a president, vice-president and secretary. How many different ways can those positions be filled?

Solution Here, n is 18 and r is 3. Using the formula:

$$nPr = \frac{18!}{15!} = \boxed{4896 \text{ possibilities}}$$

TRY IT Your iPod contains 954 songs and you want the iPod to pick 5 songs at random. Assuming songs cannot be repeated, how many 5 song play lists can your iPod generate?

Notice that if n and r are the same, like when we arranged 7 textbooks in 7 positions, the formula becomes

$$\frac{7!}{0!}$$

Note: $0! = 1$

and that should be equal to 7!, so 0! is defined to be equal to 1.

Combinations

So far we've considered the situation where we chose r items out of n possibilities without replacement and where the order of selection was important. We now focus on a similar situation in which the order of selection is not important.

EXAMPLE 7 **COMBINATIONS**

A charity benefit is attended by 25 people at which three $50 gift certificates are given away as door prizes. Assuming no person receives more than one prize, how many different ways can the gift certificates be awarded?

Solution Using the Basic Counting Rule, there are 25 choices for the first person, 24 remaining choices for the second person and 23 for the third, so there are $25 \cdot 24 \cdot 23 = 13,800$ ways to choose three people. Suppose for a moment that Abe is chosen first, Bea second and Cindy third; this is one of the 13,800 possible outcomes. Another way to award the prizes would be to choose Abe first, Cindy second and Bea third; this is another of the 13,800 possible outcomes.

But either way Abe, Bea and Cindy each get $50, so it doesn't really matter the order in which we select them. In how many different orders can Abe, Bea and Cindy be selected? It turns out there are 6:

$$ABC \qquad ACB \qquad BAC \qquad BCA \qquad CAB \qquad CBA$$

How can we be sure that we have counted them all? We are really just choosing 3 people out of 3, so there are $3 \cdot 2 \cdot 1 = 6$ ways to do this; we didn't really need to list them all; we can just use permutations!

So, out of the 13,800 ways to select 3 people out of 25, six of them involve Abe, Bea and Cindy. The same argument works for any other group of three people (say Abe, Bea and David or Frank, Gloria and Hildy) so each three-person group is counted six times. Thus the 13,800 figure is six times too big. The number of distinct three-person groups will be

$$\frac{13,800}{6} = \boxed{2300 \text{ groups}}$$

It turns out that we can generalize this rule, and if order doesn't matter, we can count the number of ways to choose r items out of n by counting the number of permutations (where order matters) and dividing by $r!$ (the number of ways to arrange the items we've chosen).

Combinations

We say that there are nCr combinations of size r that may be selected from among n choices without replacement when **order does not matter**. To compute the number of combinations of r items from a collection of n items, we use the formula below:

$$nCr = \frac{n!}{(n-r)!r!}$$

Thus, the only distinction between *permutations* and *combinations* is whether or not order matters. In questions that don't explicitly state whether to use the permutation or combination formula, all we have to consider is whether or not order is important in that scenario. For instance, when picking a president, vice president, and secretary for a club, order matters, but when picking a committee without ranks, order doesn't matter; all that matters is who got chosen.

EXAMPLE 8 STUDENT COUNCIL

A group of four students is to be chosen from a 35-member class to represent the class on the student council. How many ways can this be done?

Solution

Simply use the combination formula with $n = 35$ and $r = 4$:

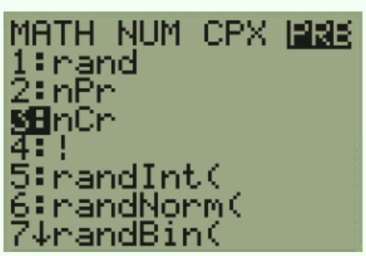
$$nCr = \frac{35!}{31!\ 4!} = \boxed{52,360 \text{ ways}}$$

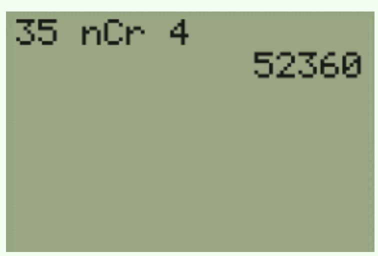
TRY IT

The United States Senate Appropriations Committee consists of 29 members; the Defense Subcommittee of the Appropriations Committee consists of 19 members. Disregarding party affiliation or any special seats on the Subcommittee, how many different 19-member subcommittees may be chosen from among the 29 Senators on the Appropriations Committee?

Using Your Calculator

Combinations can also be calculated using a graphing calculator. To do so, select the `nCr` operation from the `PRB` (probability) menu.





We'll show two more examples, but each of these is simply a matter of applying the correct formula in the right way.

Again, to distinguish between permutations (`nPr`) and combinations (`nCr`), simply ask, "is order significant in this problem?" If it is, that's a permutation, and if not, it is a combination.

EXAMPLE 9

QUIDDITCH PLAYERS

How many different ways can Harry Potter choose 3 players to be "Chasers" from a choice of 10 players?

Solution

Does the order of choice matter? Since all 3 players will fill the same position, there's no difference in what order they are selected, so this is a combination problem:

$$10\ nCr\ 3 = \frac{10!}{7!\ 3!} = \boxed{120 \text{ ways}}$$

TRY IT

How many different four-card hands can be dealt from a deck that has 16 different cards?

EXAMPLE 10

ATTENDING A WORKSHOP

How many different ways can a director select 4 actors from a group of 20 actors to attend a workshop on performing in rock musicals?

Solution

Does the order matter? No, there is no mention of any distinction based on what order the actors are selected; all that matters is which ones were chosen or not chosen. Thus, this too is a combination problem.

$$20\ nCr\ 4 = \frac{20!}{16!\ 4!} = \boxed{4845 \text{ ways}}$$

TRY IT

You volunteer to pet-sit for your friend who has seven different animals. You offer to take three of the seven. How many different sets of pets can you care for?

Probability and the Counting Methods

We can use permutations and combinations to help us answer more complex probability questions than we could before, specifically by allowing us to more efficiently count the number of possible outcomes in our sample space and those that correspond to our event.

EXAMPLE 11

BIC PENS

Bic Pens make pens in 4 colors–blue, black, red and green–with 3 tip styles: extra fine, fine and medium. What is the probability of picking one pen at random and having it be a black pen?

Solution

For this example, we just need the fundamental counting principle; since there are 4 colors and 3 tips, there are a total of 12 pen styles.

If we consider black pens, we have 1 color and 3 tips, so there are 3 choices for black pens.

Therefore the probability of picking one pen at random and having it be black is:

$$P(\text{black pen}) = \boxed{\frac{3}{12} = \frac{1}{4}}$$

LOTTERY — EXAMPLE 12

In a certain state's lottery, 48 balls numbered 1 through 48 are placed in a machine and six of them are drawn at random, without replacement. If the six numbers drawn match the numbers that a player had chosen, the player wins \$1,000,000. In this lottery, the order in which the numbers are drawn doesn't matter. Compute the probability that you win the million-dollar prize if you purchase a single lottery ticket.

Solution

Here, we'll use what we know about combinations, since we're picking 6 numbers out of 48 without worrying about the order in which they're arranged.

There are

$$nCr = \frac{48!}{42!\ 6!} = 12,271,512$$

ways to choose six numbers, and only one winning combination, so the probability of winning is

$$\boxed{\frac{1}{12,271,512} \approx 0.00000008}$$

TRY IT

Suppose you play the Daily Pick 3 everyday, in which three numbers are chosen without replacement, and choose the numbers 214. You play \$1 straight and \$1 boxed. (Straight means you pick three numbers and in order to win, you must have those three numbers in the exact same order. Boxed means you win with any permutation of those three numbers.) What is the probability of you winning the Daily Pick 3 straight? What is the probability of you winning the Daily Pick 3 boxed?

PIN — EXAMPLE 13

A 4-digit PIN is selected. What is the probability that there are no repeated digits?

Solution

There are a total of

$$\underline{10} \cdot \underline{10} \cdot \underline{10} \cdot \underline{10} = 10,000$$

possible PINs, so all we need to find is how many of them have no repeating digits.

This is equivalent to choosing 4 digits out of the 10 possible digits (so that we get a distinct value in each of the 4 positions), regardless of order, meaning that there are

$$nCr = \frac{10!}{6!\ 4!} = 210$$

possible PIN codes with no repeating digits. The probability of this occurring, then, is

$$\boxed{\frac{210}{10,000} = 0.021}$$

The principle is simple: count the number of total possibilities and count how many ways your event can occur, then divide these two. In practice, though, it can be tricky to accomplish this counting. Let's go back to a deck of cards for another example.

EXAMPLE 14 DRAWING CARDS

Compute the probability of randomly drawing five cards from a deck without replacement and getting exactly one Ace.

Solution

Getting *exactly* one Ace means getting one Ace and four cards that are not Aces (this seems obvious, but stating it is helpful in order to see the next step): we need to find the number of ways to pick one Ace and the number of ways to pick 4 non-Aces, and then multiply them (remember the fundamental counting principle). Note that order doesn't matter, so we'll use the *combination* formula.

$$\begin{aligned}&\text{ways to pick one ace } \times \text{ ways to pick 4 non-aces}\\&={}_4C_1 \times {}_{48}C_4\\&= 4 \times 194,580 = 778,320\end{aligned}$$

Now, to compute the probability of this occurring, we need to divide the number of ways it could happen by the total number of possible hands: ${}_{52}C_5 = 2,598,960$

$$\boxed{\frac{778,320}{2,598,960} \approx 0.2995}$$

TRY IT

Compute the probability of randomly drawing five cards from a deck of cards and getting three Aces and two Kings.

EXAMPLE 15 MOVIE ORDER

Assume that you are still in charge of the FCC movie marathon. Recall the four types of movies you were planning on showing are Action, Comedy, Drama and Horror. Assume someone also brought a Thriller to be included in the marathon. In order to find out the order of the movies, you decide to throw all the names in a hat and plan to draw one name out of the hat at a time. The order will be determined by how they are pulled out of the hat. What is the probability of the Thriller being played 4th and the Horror movie being played 5th?

Solution

1. How many ways can this scenario happen?

   We can do this two ways: using permutations or thinking through the fundamental counting principle; since the order of the fourth and fifth movies is given, all that remains to be ordered is the first three.

   a) Using permutations: ${}_3P_3 = 6$

   b) Using the fundamental counting principle:

   $$\underline{3} \cdot \underline{2} \cdot \underline{1} \cdot \underline{1} \cdot \underline{1} = 6$$

2. How many options are there for the order of the movies?

   There are 5 movies being organized in 5 slots:

   $${}_5P_5 = 120$$

3. What is the probability of this scenario?

   The probability is simply the number of ways the scenario could occur divided by the number of total possibilities:

   $$\boxed{\frac{6}{120} = 0.05}$$

Exercises 4.4

In problems 1–4, use your calculator to evaluate the given expression.

1. $12!$

2. $\frac{5!}{3!}$

3. ${}_8P_3$

4. ${}_{22}C_4$

5. A license plate is to have the following form: three letters followed by three numbers. An example of a license plate would be MTH 314. How many different license plates can be made, assuming that letters and numbers can be reused?

6. A bride is choosing a dress for her bridesmaids. If there are 4 styles, 3 colors, and 6 fabrics, in how many ways can she select the bridesmaids' dress?

7. A quiz consists of 5 true-or-false questions. In how many ways can a student answer the quiz?

8. A boy owns 2 pairs of pants, 3 shirts, 8 ties, and 2 jackets. How many different outfits can he wear to school if he must wear one of each item?

9. Lisa is shopping for a new car. She has the following decisions to make: type (sedan, SUV, pick-up truck), make (domestic or import), color (black, white, silver). In how many ways can she select her new vehicle?

10. A social security number contains nine digits, such as 999-04-6756. How many different social security numbers can be formed? Do you think we will ever run out?

11. How many different ways can the letters of the word MATH be rearranged to form a four-letter code word?

12. How many ways can we select five door prizes from seven different ones and distribute them among five people?

13. Eight sprinters have made it to the Olympic finals in the 100-meter race. In how many different ways can the gold, silver, and bronze medals be awarded?

14. At a charity benefit with 25 people in attendance, three $50 gift certificates are given away as door prizes. Assuming no person receives more than one prize, how many different ways can the gift certificates be awarded?

15. Ten bands are to perform at a weekend festival. How many different ways are there to schedule their appearances?

16. There are seven books in the Harry Potter series. In how many ways can you arrange the books on your shelf?

17. How many different four-card hands can be dealt from a 52 card deck?

18. There are 40 runners in a race, and no ties. In how many ways can the first three finishers be chosen from the 40 runners, regardless of how they are arranged?

19. There are 24 students in a MATH 120 class at FCC. How many ways are there to select 5 students for a group project?

20. A local children's center has 55 kids, and 6 are selected to take a picture for the center's advertisement. How many ways are there to select 6 children for the picture?

21. In a lottery game, a player picks six numbers from 1 to 48. If exactly 4 of the 6 numbers match those drawn, the player wins third prize. What is the probability of winning this prize?

22. Compute the probability of randomly drawing five cards from a deck and getting 3 Aces and 2 Kings.

23. You own 16 CDs. You want to randomly arrange 5 of them in a CD rack. What is the probability that the rack ends up in alphabetical order?

24. A jury pool consists of 27 people, 14 men and 13 women. Compute the probability that a randomly selected jury of 12 people is all male.

25. Compute the probability that a 5-card poker hand is dealt to you that contains all hearts.

26. Compute the probability that a 5-card poker hand is dealt to you that contains four Aces.

27. A jury pool has 18 men and 21 women, from which 12 jurors will be selected. Assuming that each person is equally likely to be selected and that the jury is selected at random, find the probability the jury consists of

(a) all men

(b) all women

(c) 8 men and 4 women

(d) 6 men and 6 women

28. A race consisted of 8 women and 10 men. What is the probability that the top 3 finishers were:

(a) all men

(b) all women

(c) 2 men and 1 woman

(d) 1 man and 2 women