Versatile Mathematics

COMMON MATHEMATICAL APPLICATIONS

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Attributions This book benefited tremendously from others who went before and freely shared their creative work. The following is a short list of those whom we have to thank for their work and their generosity in contributing to the free and open sharing of knowledge.

- David Lippman, author of *Math in Society*. This book uses sections derived from his chapters on Finance, Growth Models, and Statistics. He also administers MyOpenMath, the free online homework portal to which the problems in this text were added.
- The developers of onlinestatbook.com.
- OpenStax College (their book *Introductory Statistics* was used as a reference)
  OpenStax College, *Introductory Statistics*. OpenStax College. 19 September 2013. <http://cnx.org/content/col11562/latest/>
- The authors of OpenIntro Statistics, which was also used as a reference.
- The Saylor Foundation Statistics Textbook: http://www.saylor.org/site/textbooks/Introductory%20Statistics.pdf

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Contents

Chapter

5 Linear Programming

After World War II, the victorious Allies divided Germany into four sectors, with one Allied nation administering each region. Berlin, located deep within East Germany, controlled by the Soviet Union, was also split into four regions, with the United States, the United Kingdom, and France controlling the western half of the city and the Soviet Union controlling the eastern half.

However, Stalin wouldn't rest until all of Germany was under Soviet control, and in 1948, in an effort to drive the other Allied forces out without declaring open war, the USSR blockaded West Berlin, cutting off road, rail, and canal lines into the city from West Germany.

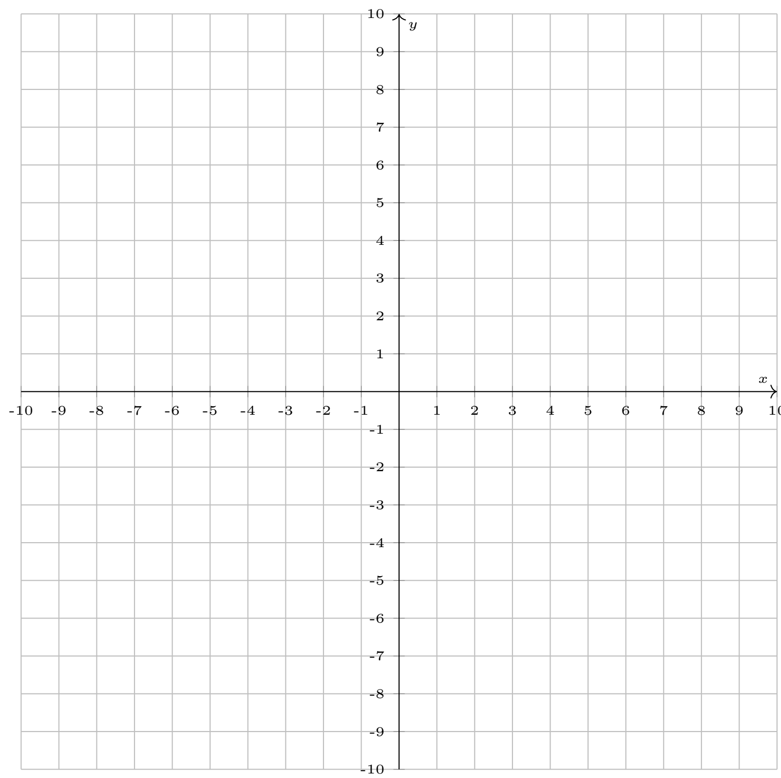
The Allies responded with an immense effort known as the Berlin Airlift, eventually moving 8,000 tons of food and fuel *per day* into West Berlin in massive cargo planes. To handle the incredibly complex logistics of this process, the Allies turned to a new area of applied mathematics known as linear programming, the topic we'll investigate in this chapter. Linear programming, part of a wider field known as *mathematical optimization*, is used today in fields ranging from business and economics to engineering and manufacturing, solving problems involving the allocation of limited resources.

SECTION 5.1 Linear Functions and Their Graphs

Before we can solve linear programming problems, we need to review linear equations and inequalities, because that is the language used to state optimization problems. We'll be given a goal and the limitations on our resources in terms of linear equations and inequalities, and we'll need to graph and solve them in order to find the optimal use of our resources.

Plotting Points

Everything starts with the rectangular, or Cartesian, coordinate plane, named for René Descartes. As the legend goes, Descartes was lying on his bed one day, watching a fly scurry across the ceiling. Out of idle curiosity, he wondered if there were a simple way to describe the position of the fly, and he realized that if he specified how far it was from two of the walls, that would clearly define its position. Whether or not the legend has any basis in reality, it illustrates the idea behind the rectangular coordinate system.



This is familiar to residents of NYC; for instance, the Metropolitan Museum of Art is located at the intersection of 5th Ave and E 82nd St

This coordinate system consists of two number lines–the x axis and the y axis–placed at a right angle to each other, crossing at the **origin**. You can think of the grid that these axes form as the map of a well-planned city, with north-south streets crossing east-west streets at consistent intervals. If you want to specify a location in the city, all you have to specify is an intersection.

This is how we plot points, by specifying their east-west location using an x-coordinate and specifying their north-south location using a y-coordinate. These are written as an ordered pair (x, y).

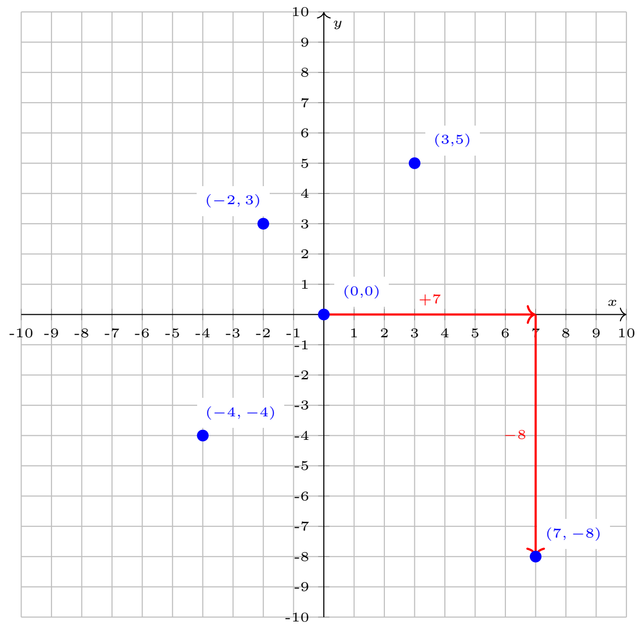
PLOTTING POINTS

EXAMPLE 1

Plot the points (0,0), (3,5), $(-2,3)$, $(-4,-4)$, and $(7,-8)$.

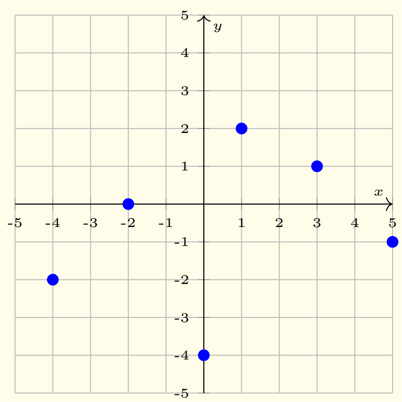
Solution

For each point, start at the origin and step to the right or left according to the first number in the ordered pair, then step up or down according to the second number (make sure to keep track of direction, based on whether each number is positive or negative).



TRY IT

What points are plotted on the coordinate system shown here?



Once we can plot points, we can work up from that to graphing lines (and if we wanted to, graphing all sorts of things). Linear graphs consist of infinitely many points that all lie along a straight line that extends forever in either direction.

Graphing Lines

Take a look at an equation like

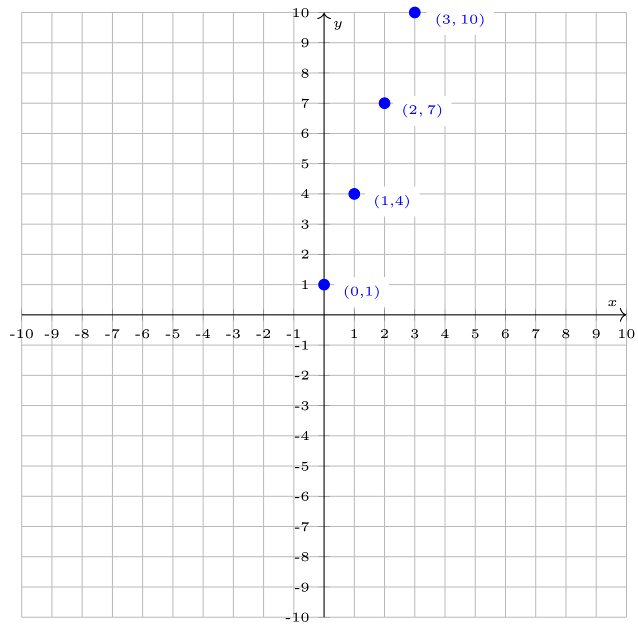
$$y = 3x + 1.$$

This equation gives a relationship between x and y; it simply says that whatever x is, there is a corresponding y that you get by multiplying x by 3 and adding 1.

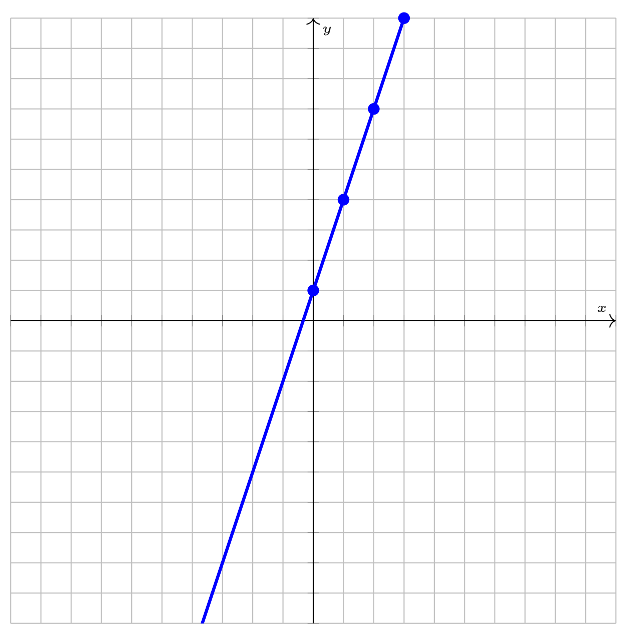
For instance,

if x is 0, y is 1
if x is 1, y is 4
if x is 2, y is 7
if x is 3, y is 10

We can write each of these as an ordered pair (x, y), and each of those corresponds to a point on the coordinate plane:



It's clear that all these points lie along a straight line, and if we picked x values between the ones we picked, we would just be filling in the points between these. Thus, we conclude that this is an example of a linear equation and we can draw the line that connects these dots.



Notice, however, that we worked harder than we had to; once we had two points, the line was decided, and the other points just fell into place along that line. This leads us to an important conclusion.

No matter how you graph a line, the entire process comes down to one simple fact:

Key to Graphing Lines

Two points determine a line.

Any time you graph a line, if all else fails, just try to find two points on it (by picking two sample values for x like we did above and finding the corresponding y values) and draw the line that passes through those two points.

Note We'll only be dealing with linear equations in this chapter, but in general, you can recognize linear equations by the fact that they only have x's and y's and constants in them, and nothing like x^2 or 2^y or $\sqrt{x}$. For example, each of the following are linear equations:

$$3x + 4y = 7$$
$$9x - 16y = -3$$
$$y = 8x - 14$$
$$x = 2y + 5$$

Graphing Lines Using Slope and Intercept

Look back at that example. The equation we started with was $y = 3x + 1$, and by picking a few x's and plugging them into the equation to find the matching y's, we got

x	y
0	1
1	4
2	7
3	10

and we could easily have kept going, or turned and starting using negative x values.

However, from this table we can notice two interesting things:

1. Each time we increase x by 1, y increases by 3. Notice that 3 is the coefficient of x in the equation. We call this the **slope** of the equation, because it describes how the line is angled. Look back at the graph and notice how, traveling from left to right, the line travels upward 3 units for every unit forward.

Note: Slope is "rise over run": $\frac{\text{rise}}{\text{run}}$

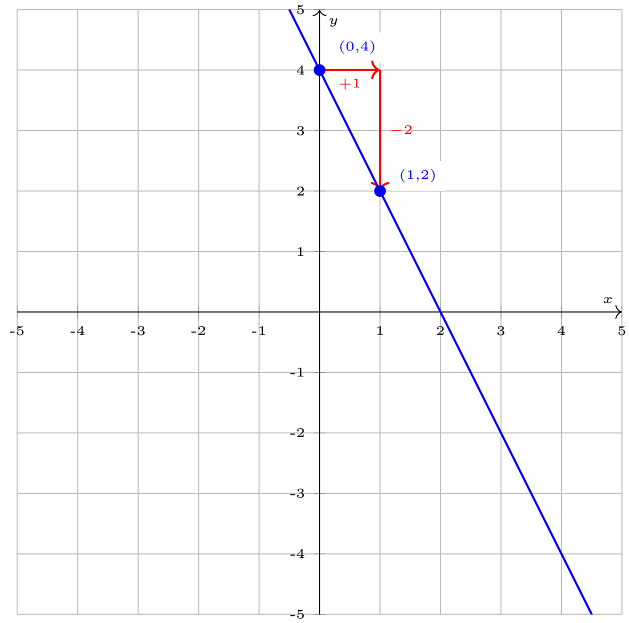
2. When x is 0, y is 1, which is the constant in the equation (this is not accidental). This is called the ***y*-intercept**, because it is the point where the line crosses the y-axis. Look back at the graph and notice how the line crosses the y-axis at 1.

This gives us a quick way to graph a line that is written in *slope-intercept form*: $y = mx + b$, where m is the slope and b is the y-intercept. It's still based on graphing by plotting two points: we can graph the intercept first, then go right one unit and up or down however many units the slope tells us to and plot a second point, then connect these two dots.

EXAMPLE 2 GRAPHING USING SLOPE AND INTERCEPT

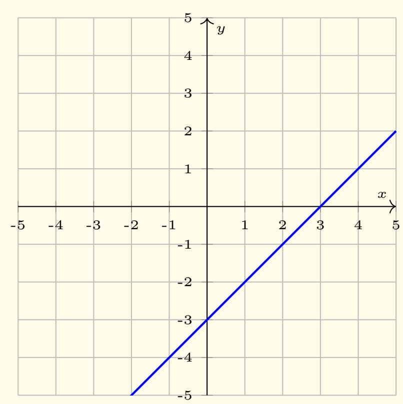
Graph the line $y = -2x + 4$.

Solution In this equation, the slope is -2 and the intercept is 4. We know then that the line crosses the y-axis at 4 and travels down two units for every one it travels to the right:



TRY IT

Find the equation of the line graphed below in slope-intercept form.

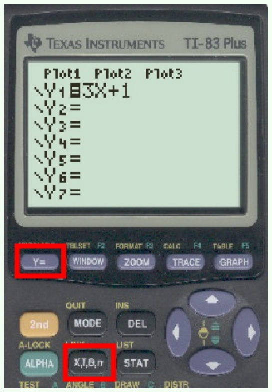


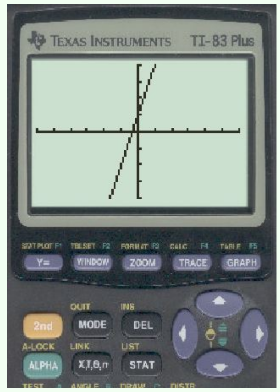
Of course, if a linear equation is not written in slope intercept form, a little algebra can manipulate it until it is:

$$\begin{aligned} 4x + 2y &= 8 \\ 2y &= -4x + 8 \\ y &= -2x + 4 \end{aligned}$$

Using Your Calculator

You can also use a graphing calculator to graph a linear equation if it is written in slope-intercept form. To do so, press the Y= button in the upper lefthand corner and enter the equation, using the X, T, θ, n button to enter x. Then press GRAPH to see the line.



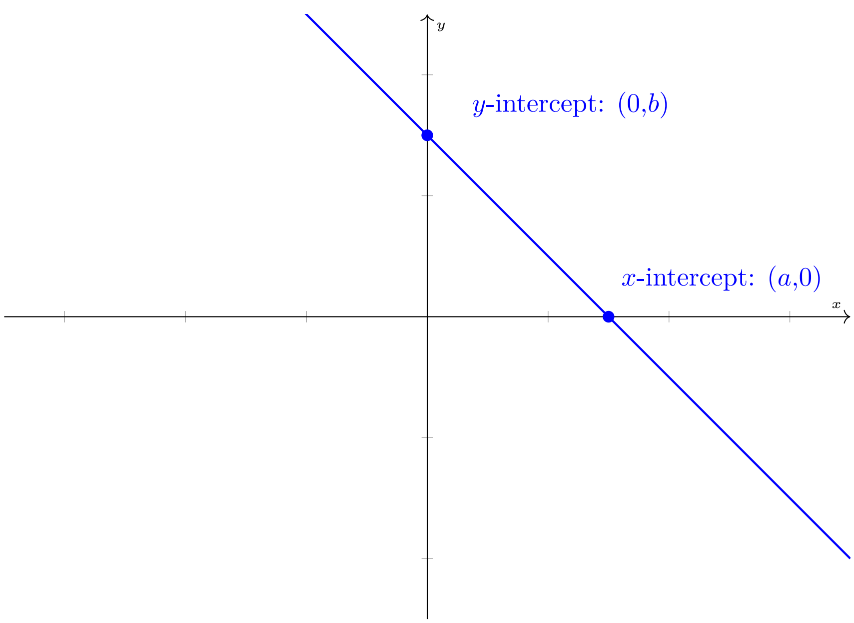


Graphing Lines Using Intercepts

Remember, whenever we graph a line, we're relying on the principle that ***two points determine a line***.

Any two points that lie on the line will do; however, in the examples we'll be doing in this chapter, it will often simplify things if we use a specific set of two points, namely the **intercepts**.

We've already seen the y-intercept, the point where the line crosses the y-axis. The x-intercept, naturally, is the point where the line crosses the x-axis.



Notice the key point about intercepts (this is how we'll find them):

- At the x-intercept, the y-coordinate is always 0
- At the y-intercept, the x-coordinate is always 0

Thus, finding the intercepts will come down to filling in the missing pieces in the following table:

x	y
0	
	0

EXAMPLE 3 GRAPHING LINES USING INTERCEPTS

Graph the line $x - 0.5y = 5$ using the intercepts.

Solution

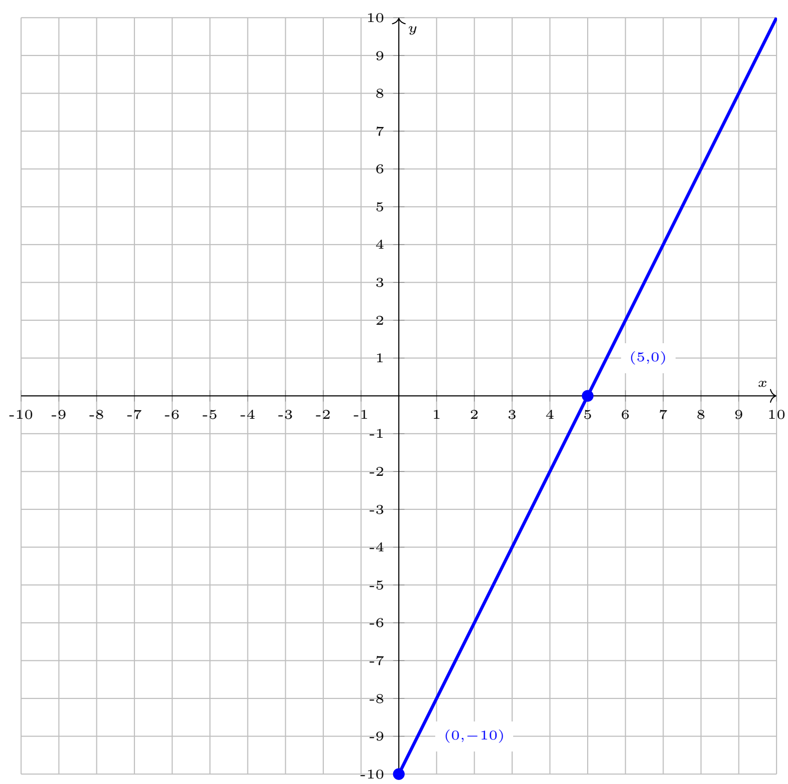
- Find the x-intercept: let $y = 0$ and find its corresponding x value:

$$x - 0.5(0) = 5 \longrightarrow x = 5$$

- Find the y-intercept: let $x = 0$ and find its corresponding y value:

$$0 - 0.5y = 5 \longrightarrow -0.5y = 5 \longrightarrow y = \frac{5}{-0.5} = -10$$

The intercepts are thus $(5, 0)$ and $(0, -10)$, and we can graph the line by plotting these two points and connecting them.



GRAPHING LINES USING INTERCEPTS

EXAMPLE 4

Graph the line $2x + y = 8$ using the intercepts.

Solution

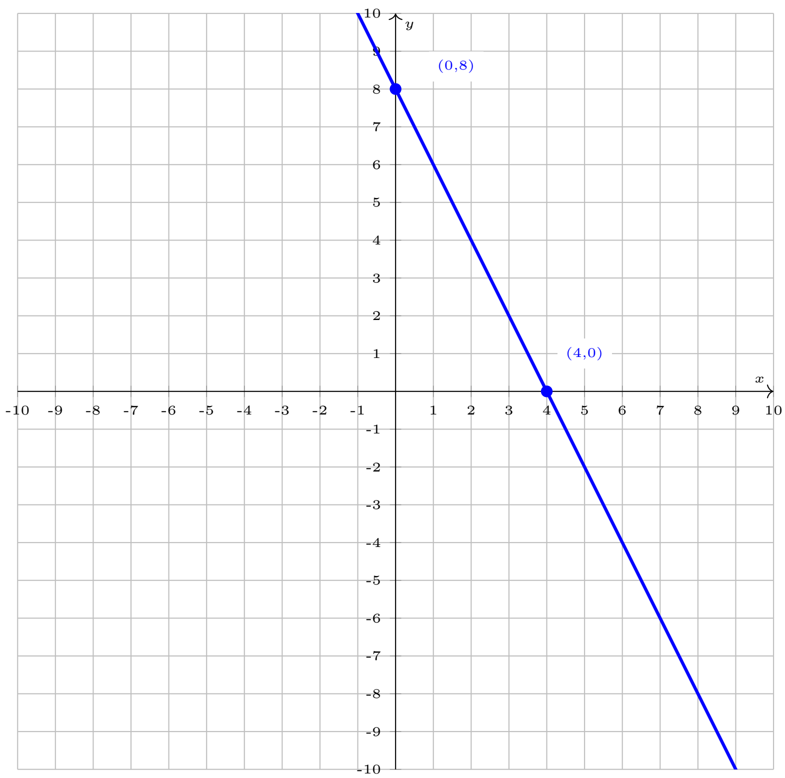
- Find the x-intercept: let $y = 0$ and find its corresponding x value:

$$2x + (0) = 8 \longrightarrow x = \frac{8}{2} = 4$$

- Find the y-intercept: let $x = 0$ and find its corresponding y value:

$$2(0) + y = 8 \longrightarrow y = 8$$

The intercepts are thus $(4, 0)$ and $(0, 8)$, and we can graph the line by plotting these two points and connecting them.



Horizontal and Vertical Lines

Lastly, we will need to be able to easily recognize and graph horizontal and vertical lines. Think for a moment for what it means for a line to be horizontal: for every x value, the y value of a point on the line is the same constant. Thus, whenever we see an equation that looks like

$$y = b$$

for any constant b, we'll know that it is a horizontal line at b. Similarly, on a vertical line, the x-coordinate is consistent for any y value, so vertical lines have the form

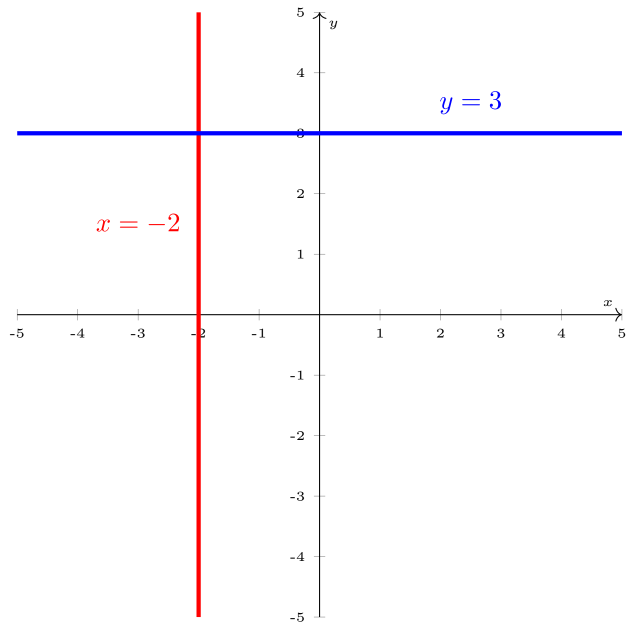
$$x = a$$

for some constant a.

EXAMPLE 5 GRAPHING HORIZONTAL AND VERTICAL LINES

Graph the lines $y = 3$ and $x = -2$.

Solution The line $y = 3$ is a horizontal one at 3, and $x = -2$ is a vertical line at -2:



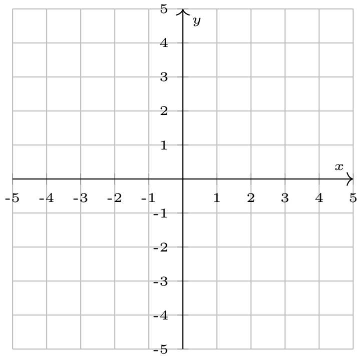
Summary: Horizontal and Vertical Lines

- Horizontal lines have the form
$$y = b.$$
- Vertical lines have the form
$$x = a.$$

Exercises 5.1

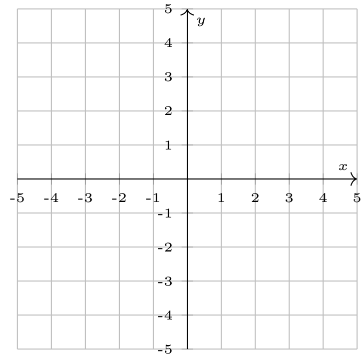
1. Plot the following points.

$$(0,3),(-4,2),(3,-2),(-1,-2)$$



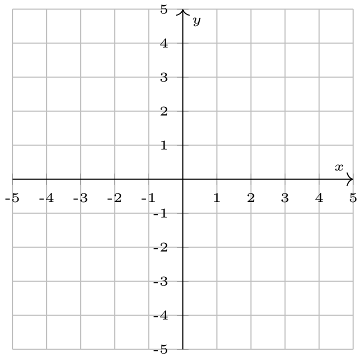
2. Plot the following points.

$$(-3,1),(4,3),(2,-2),(1,-4)$$



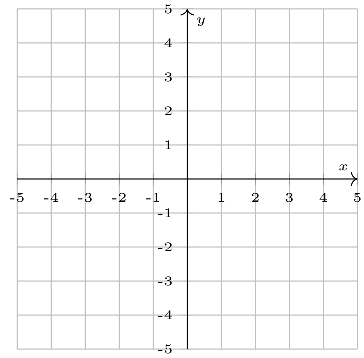
3. Graph the following line using the intercepts.

$$2x+5y=10$$



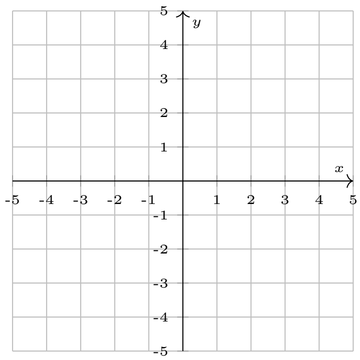
4. Graph the following line using the intercepts.

$$4x+3y=12$$



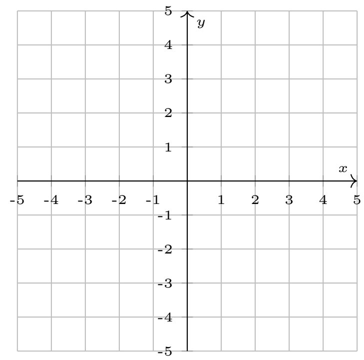
5. Graph the following line using the intercepts.

$$4x+4y=8$$



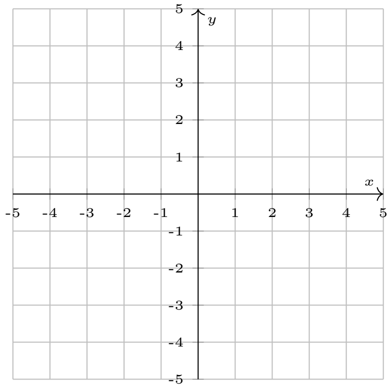
6. Graph the following line using the intercepts.

$$x+3y=3$$



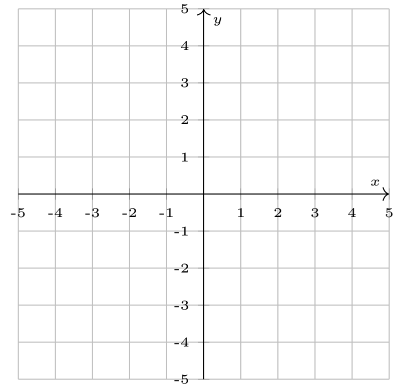
7. Graph the following line using the slope and y-intercept.

$$y = 2x - 3$$



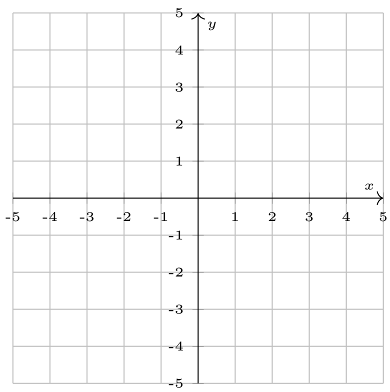
8. Graph the following line using the slope and y-intercept.

$$y = -x + 1$$



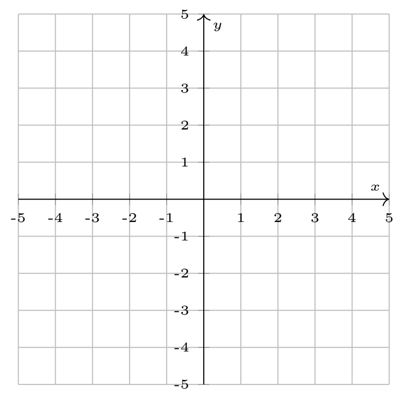
9. Graph the following line using the slope and y-intercept.

$$y = \frac{1}{2}x + 4$$



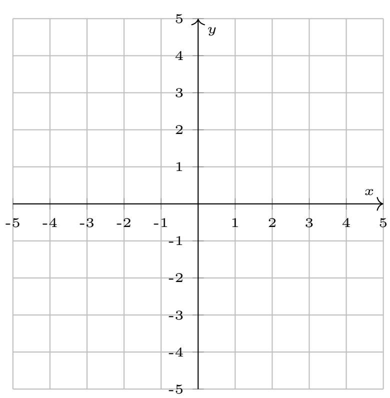
10. Graph the following line using the slope and y-intercept.

$$x - 3y = 9$$



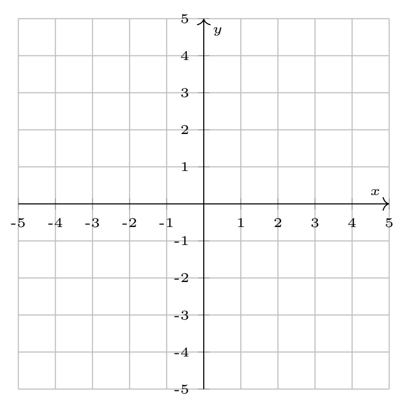
11. Graph the following line using any method.

$$x + y = 4$$



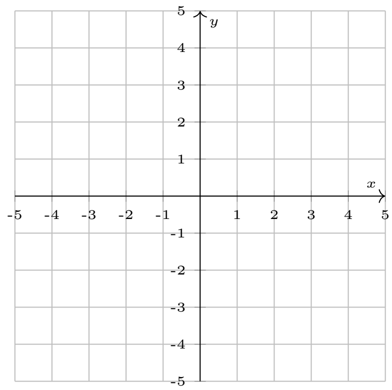
12. Graph the following line using any method.

$$-2x + 3y = 6$$



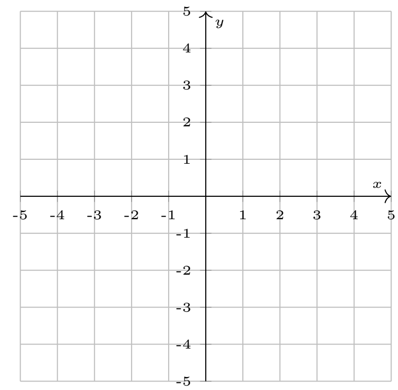
13. Graph the following line using any method.

$$x = -3$$



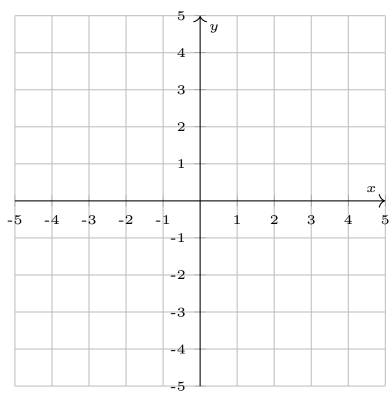
14. Graph the following line using any method.

$$2y + x = 4$$



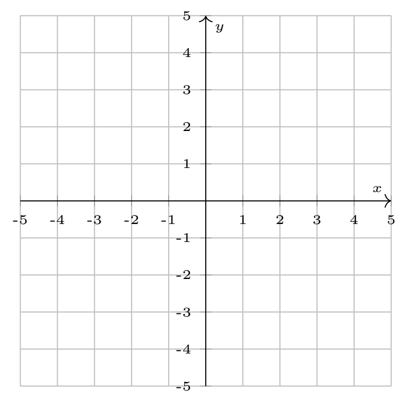
15. Graph the following line using any method.

$$y = 2$$



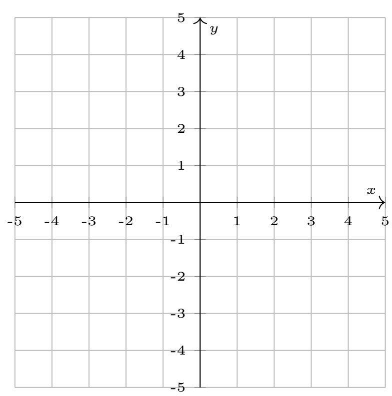
16. Graph the following line using any method.

$$3x + 2y = 4$$



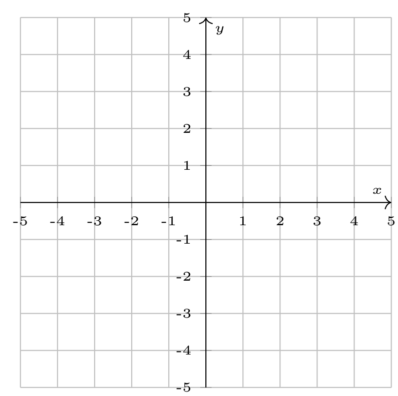
17. Graph the following line using any method.

$$-5x - 4y = 12$$



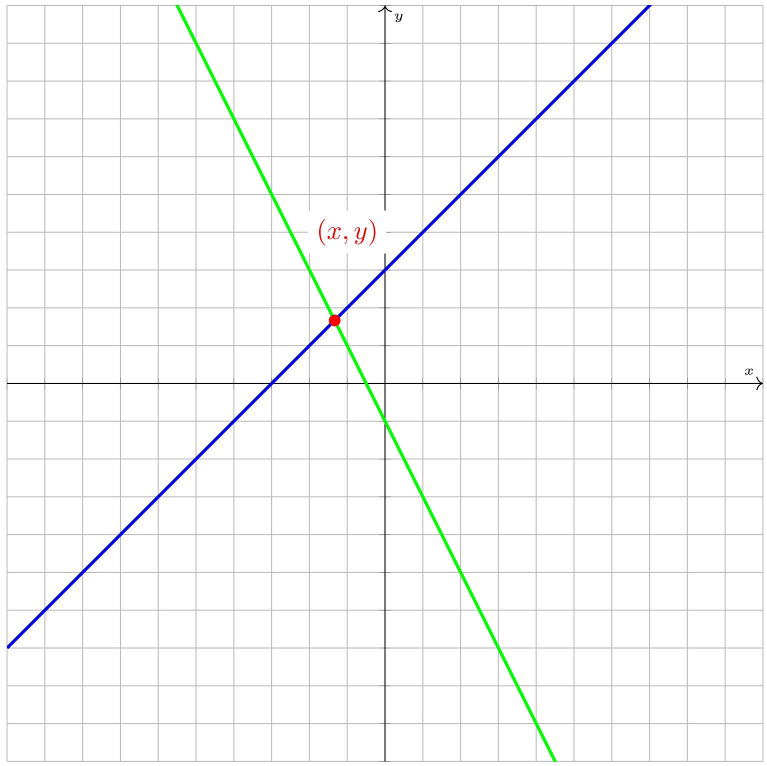
18. Graph the following line using any method.

$$y = -3x + 5$$



SECTION 5.2 Systems of Linear Equations

Later in this chapter, we'll want to find where two lines intersect.



When we solve this, what we are finding is the point (x, y) that lies on both lines.

Solving a System of Linear Equations

Solving a system of linear equations means finding an x, y pair that fits both equations at once.

To see this, take the system of equations (pair of lines) shown below.

$$2x + y = 5$$
$$x - 3y = -8$$

We can show that the point $(1, 3)$ is the point where they cross by substituting it into both equations and seeing that it fits into both of them:

$$2(1) + (3) = 5 \quad \text{TRUE}$$
$$(1) - 3(3) = -8 \quad \text{TRUE}$$

You can check other points by substituting them into these two equations, but you won't find any other combination of x and y that satisfies the system. For instance, $(2, 1)$ is a solution to the first equation, but not to the second.

That illustrates how we can check a proposed solution (intersection point of two lines), but it doesn't show how to *find* the solution. In this section, we'll see three methods for solving systems like this: one graphical and two algebraic.

Solving by Graphing

If we can graph the two lines and simply spot where they cross, we can check our answer by substituting it into the system.

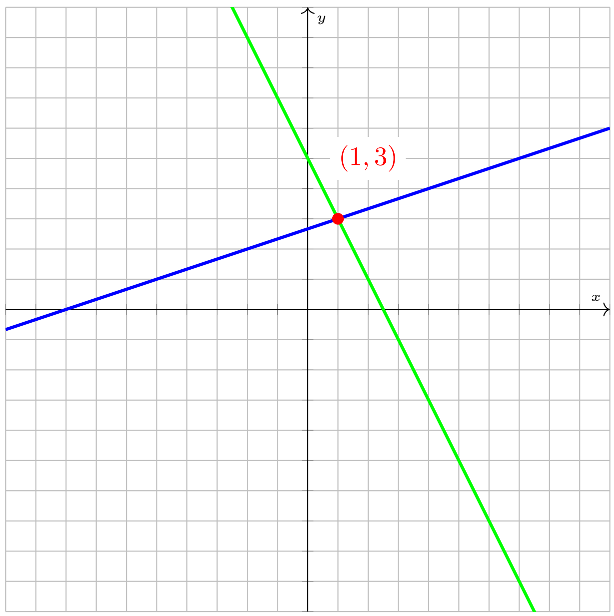
SOLVING A LINEAR SYSTEM BY GRAPHING — EXAMPLE 1

Solve the following system of equations by graphing.

$$\begin{aligned} 2x + y &= 5 \\ x - 3y &= -8 \end{aligned}$$

Solution

Start by graphing the two lines, using one of the methods in the previous section.



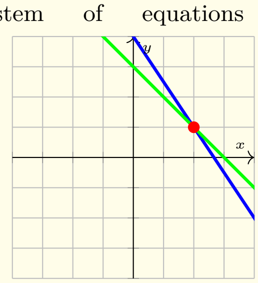
It *looks* like the lines cross at $(1, 3)$, but maybe our lines aren't perfectly graphed, and maybe they cross at $(1.01, 2.99)$. To make sure that they cross at exactly $(1, 3)$, we can substitute this point into both equations and make sure it is a solution to both.

$$\begin{aligned} 2(1) + (3) &= 5 \quad \text{TRUE} \\ (1) - 3(3) &= -8 \quad \text{TRUE} \end{aligned}$$

TRY IT

Find the solution of the system of equations graphed below.

$$\begin{aligned} 3x + 2y &= 8 \\ 2x + 2y &= 6 \end{aligned}$$



As we noted, solving by graphing only works when we can easily spot the solution, and it happens to be nice round numbers. In more general problems, we'll need to solve systems like this algebraically rather than graphically.

We have to algebraic ways to solve a system of linear equations, but both boil down to reducing the system of equations to a single equation with only one unknown–either x or y. Once we have that, we can solve the resulting equation and get half of the solution–one of the coordinates of the intersection–and use that half to get the other half.

Solving by Substitution

The first way that we'll reduce a system of two equations with two unknowns to a single equation with one unknown is by **substitution**.

Look at the system of equations below.

$$2x + 3y = 1$$
$$-2x + y = 3$$

Just focus on the first equation for a moment. If we knew what y was, we could substitute that into the first equation and get an equation that only involved x, which we could solve. Unfortunately, we don't have a number for y yet, but we *do* know that y is equal to $2x + 3$. How do we know this? This piece of information comes from the previously ignored second equation:

$$-\,2x + y = 3$$
$$+\,2x \qquad\quad +\,2x$$

$$y = 2x + 3$$

Notice that if we substitute this *expression* into the first equation, we'll have accomplished our goal of reducing the system to a single equation with one unknown (in this case, x). Thus, simply replace y in the first equation with $2x + 3$:

$$2x + 3(2x + 3) = 1$$

This is an equation we can solve, and if we do, we find that $x = -1$.

Here now is half the answer, the x-coordinate of the point where these two lines cross, but how do we find the y-coordinate? Since this point lies on both lines, we can use either equation to find the y that corresponds to this x, but we'll use the second equation, since we've done a little work to write it as $y = 2x + 3$, which makes it easy to find the y that corresponds to $x = -1$:

$$y = 2(-1) + 3 = 1$$

The answer, then, is $x = -1$, $y = 1$, which means that the two lines cross at the point $(-1, 1)$.

Note that if we had plugged $x = -1$ into the first equation, we would have gotten the same result for y. If we didn't, it would mean we had made a mistake in finding x.

This example gives a blueprint for how to solve systems of linear equations by substitution.

Solving a Linear System by Substitution

1. Solve one of the equations for one of the variables. (Pick the easier one)
2. Substitute that expression into the *other* equation. (Substituting into the same equation won't do anything)
3. Solve the resulting equation for the variable that is left.
4. Substitute that half of the answer into *either* of the original equations to find the other half. (It's easier to substitute it into the one that you have solved for the other variable)

Notice that this method gives you a lot of choice. You can pick *either* equation to solve for *either* variable, and you can substitute the first half of the answer into *either* equation to find the other half. This lets you save some work if you look for one of the equations that is easier than the other to solve for one of the variables.

SOLVING BY SUBSTITUTION — EXAMPLE 2

Solve the following system of equations by substitution.

$$\begin{aligned} 2x + 3y &= 11 \\ x - 4y &= 0 \end{aligned}$$

Solution

It looks like the second equation is easier to solve for x, since x only has a coefficient of 1 there. Doing so gives

$$x = 4y.$$

We can now take this and plug $4y$ in for x in the first equation:

$$2(4y) + 3y = 11 \longrightarrow 11y = 11 \longrightarrow y = 1$$

We've now got half of the answer, and we can substitute 1 for y into either equation to find x, but we choose to use the rearranged form of the second equation:

$$x = 4(1) = 4$$

The solution, or the point where the two lines cross, is therefore $\boxed{(4, 1)}$.

TRY IT

Use substitution to solve the following system of equations.

$$\begin{aligned} 4x - y &= 5 \\ 3x + 3y &= 15 \end{aligned}$$

SOLVING BY SUBSTITUTION — EXAMPLE 3

Use substitution to solve the following system of equations.

$$\begin{aligned} -4x + y &= -11 \\ 2x - 3y &= 5 \end{aligned}$$

Solution

We spot a lone y in the first equation, so we solve the first equation for y and get

$$y = 4x - 11.$$

Substituting this into the second equation:

$$2x - 3(4x - 11) = 5 \longrightarrow -10x = -28 \longrightarrow x = \frac{14}{5}$$

We can plug this half of the answer into the equation $y = 4x - 11$:

$$y = 4\left(\frac{14}{5}\right) - 11 = \frac{1}{5}$$

The point where the two lines cross is

$$\boxed{\left(\frac{14}{5}, \frac{1}{5}\right)}$$

The answer in the last example is a clear instance where we could never have solved by graphing, at least by hand.

Solving by Elimination

The other algebraic method we have to solve a linear system is called **elimination**. The basic goal is still the same: to reduce the system to a single equation with one unknown. What changes, though, is how we accomplish this. We can use either substitution or elimination to solve any system of equations, but depending on the numbers used in a particular example, one may be easier than the other.

To illustrate solving by elimination, look at the following system of equations.

$$\begin{aligned} 3x + 2y &= 14 \\ 3x - 2y &= 10 \end{aligned}$$

The method of elimination uses the fact that we can add anything we want to one side of an equation as long as we add the same thing to the other side.

How does this help us? Notice that the second equation says that $3x - 2y$ and 10 are the same, so we can add $3x - 2y$ to the left side of the *first* equation and 10 to its right side. When we do so, the $2y$ and $-2y$ cancel each other:

$$\begin{aligned} 3x + 2y &= 14 \\ +3x - 2y &= 10 \\ \hline 6x &= 24 \end{aligned}$$

Thus, $x = 4$, and substituting that into either one of the equations finds that $y = 1$.

In that example, adding the equations made one of the variables cancel itself out, specifically because we had $2y$ in one equation and $-2y$ in the other. What if the equations we're working with are not so compliant?

Take, for example, the system below.

$$\begin{aligned} 2x - 7y &= 2 \\ 3x + y &= -20 \end{aligned}$$

Here, neither variable lines up so nicely, ready to be canceled, but if we multiply every term in the second equation by 7, the y's will be ready to cancel. Remember that we can multiply anything we want on one side of an equation as long as we multiply the same thing on the other side, so we're allowed to multiply the entire equation by 7, knowing that all we're changing is its form, not the actual equation.

If we do this, we get

$$\begin{aligned} 2x - 7y &= 2 \\ 21x + 7y &= -140 \end{aligned}$$

Now, when we add the two equations, y is canceled, leaving

$$23x = -138$$

which we can solve to find that $x = -6$. Finally, substituting that into either equation, we find that $y = -2$.

This leads to the following steps for solving a linear system by elimination.

Solving a Linear System by Elimination

1. If necessary, arrange the equations so that the x's and y's line up vertically.
2. If necessary, multiply one or both of the equations by some constant that will make the coefficients of one of the variables opposites (same magnitude, but one positive and one negative).
3. Add the equations, canceling one of the variables.
4. Solve the resulting equation for the variable that is left.
5. Substitute that half of the answer into *either* of the original equations to find the other half.

Notice that the last two steps are identical to the last two steps in the process of solving by substitution.

SOLVING BY ELIMINATION

EXAMPLE 4

Solve the following system of equations by elimination.

$$\begin{aligned} 2x + 3y &= -16 \\ 5x - 10y &= 30 \end{aligned}$$

Solution

Rather than multiplying one of the equations by a fraction in order to make the coefficients of x or y opposites, we'll multiply *both* equations by some value.
The simplest way to do this is to start by picking which variable we want to eliminate (notice that the coefficients of y already have opposite signs, so we'll choose to eliminate x). Next, multiply each equation by the magnitude of the *other* equation's coefficient of that variable (so we will multiply the first equation by 10 and the second equation by 3):

$$\begin{aligned} &10 \times (2x + 3y = -16) \\ &3 \times (5x - 10y = 30) \end{aligned}$$

This rewrites the system in such a way that when we add the two equations, y will be eliminated:

$$\begin{aligned} 20x + 30y &= -160 \\ 15x - 30y &= 90 \end{aligned}$$

When we add them, we get

$$35x = -70,$$

so $x = -2$. Substituting that into either equation yields that $y = -4$.

Thus, the intersection point for these two lines is $\boxed{(-2, -4)}$.

TRY IT

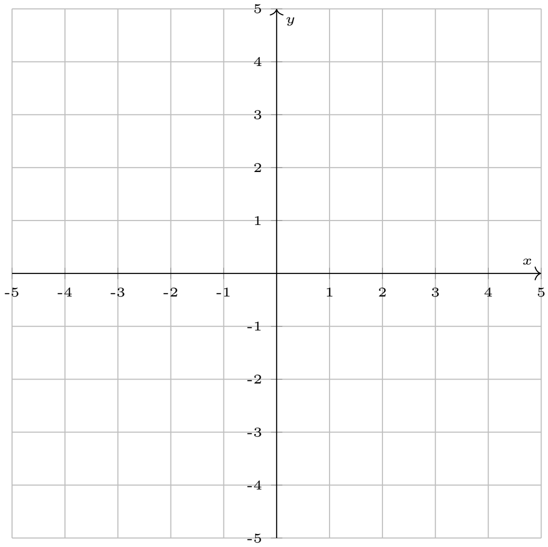
Use elimination to solve the following system of equations.

$$\begin{aligned} 4x - 3y &= -13 \\ -3x - 2y &= -3 \end{aligned}$$

Exercises 5.2

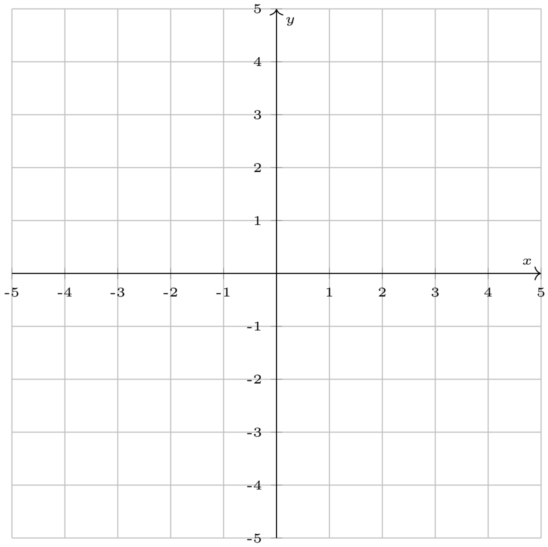
1. Solve the following system of equations by graphing.

$$\begin{aligned} 2x + y &= 4 \\ y &= 2x \end{aligned}$$



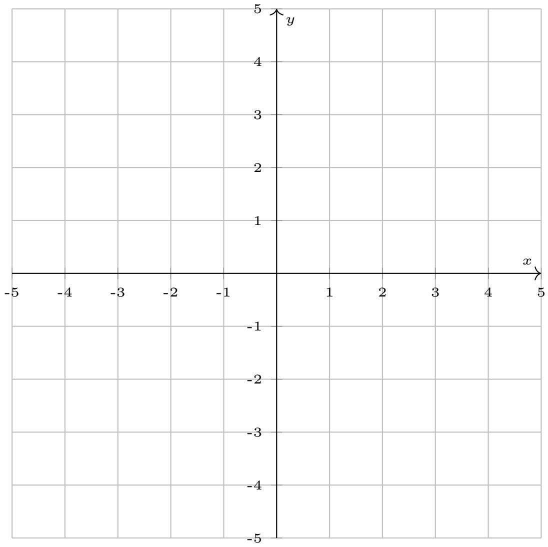
2. Solve the following system of equations by graphing.

$$\begin{aligned} 3x - 2y &= 2 \\ 4x + y &= 10 \end{aligned}$$



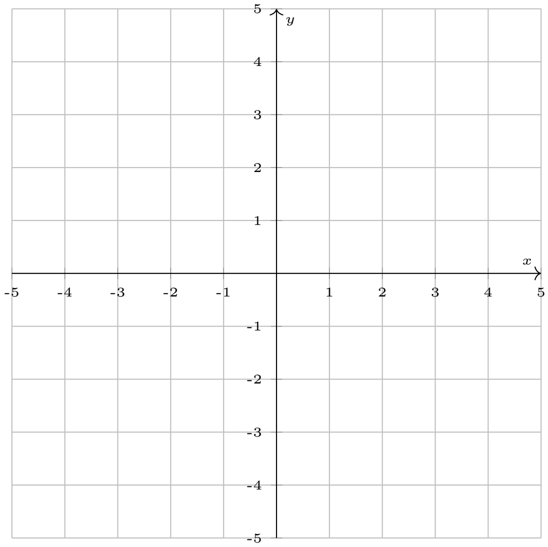
3. Solve the following system of equations by graphing.

$$\begin{aligned} -4 &= 2x - y \\ x &= 2y + 1 \end{aligned}$$



4. Solve the following system of equations by graphing.

$$\begin{aligned} x - 3y &= -6 \\ x &= -3 \end{aligned}$$



In problems 5–13, solve each system of equations by substitution or elimination.

5.

$$\begin{aligned} 3x + 5y &= -12 \\ x + 2y &= -6 \end{aligned}$$

6.

$$\begin{aligned} x - y &= 15 \\ y &= -4x \end{aligned}$$

7.

$$\begin{aligned} x + 2y &= 13 \\ y + 7 &= 4x \end{aligned}$$

8.

$$\begin{aligned} x - 3y &= 0 \\ 2x - 3y &= 6 \end{aligned}$$

9.

$$\begin{aligned} x + y &= 3 \\ x - y &= 7 \end{aligned}$$

10.

$$\begin{aligned} 3x + y &= -3 \\ 4x + y &= -4 \end{aligned}$$

11.

$$\begin{aligned} 2x + y &= -2 \\ 5x + 3y &= -6 \end{aligned}$$

12.

$$\begin{aligned} 5x + 2y &= -1 \\ 4x - 5y &= -14 \end{aligned}$$

13.

$$\begin{aligned} y &= -3x + 7 \\ 4x + 2y &= 11 \end{aligned}$$

SECTION 5.3 Systems of Linear Inequalities

In the application problems that we'll do later in this chapter, we'll come across linear *inequalities*, in addition to linear equations. Linear inequalities look like the following:

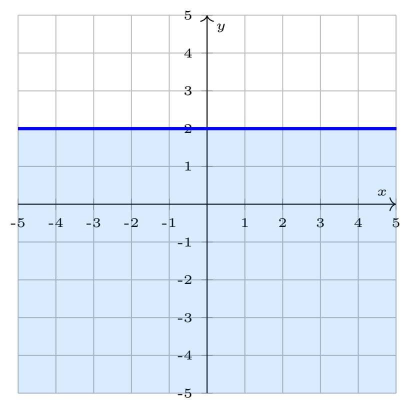
$$2x + 9y \leq 3.$$

Let's think about what this says. This says that whatever x and y are, if we multiply x by 2, multiply y by 9, and add them together, our answer will be smaller than 3 (or equal to 3). Clearly, this isn't true for *any* choice of x and y (for example, what if $x = 100$ and $y = 100$?), but the combinations of x and y that fit this inequality are called its **solution set**, just like the solution set to the equations we saw in the last two sections is the combination of x and y that fit into it and make it true.

When we graphed a linear equation, we found that all the solutions were arranged neatly along a straight line. We'd like to have a graphical interpretation for the solutions to a linear inequality, as well. Think of the following simple example:

$$y \leq 2$$

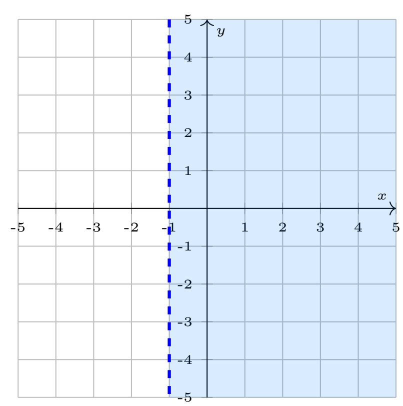
This says that all the points whose y-coordinates are 6 or less will be solutions. If we start plotting all those points, we find that they are all the points that lie below the line $y = 2$.



What about another example?

$$x > -1$$

Here, the solution set is all the points whose x-coordinate is greater than -1. However, notice that we are *not* including the points whose x-coordinate is equal to -1 (along the line $x = -1$), so we draw the line dashed this time:



These two examples illustrate how we will draw the solution set of a linear inequality. We can graph a linear inequality by first changing the inequality sign to an equals sign and graphing the resulting line.

The Solution Set of a Linear Inequality

The graph of the solutions to a linear inequality consist of every point to one side of the corresponding line.

If the inequality is $\geq$ or $\leq$, we draw the boundary solid, and if the inequality is $>$ or $<$, we draw the boundary dashed.

Once we've graphed the corresponding line, all we have to do is figure out which side of the line to shade. The simplest way to do this is to pick a test point on one side of the line and see if it is a solution. If it is, shade that side; if not, shade the other side.

GRAPHING A LINEAR INEQUALITY — EXAMPLE 1

Graph the solution set for the following inequality.

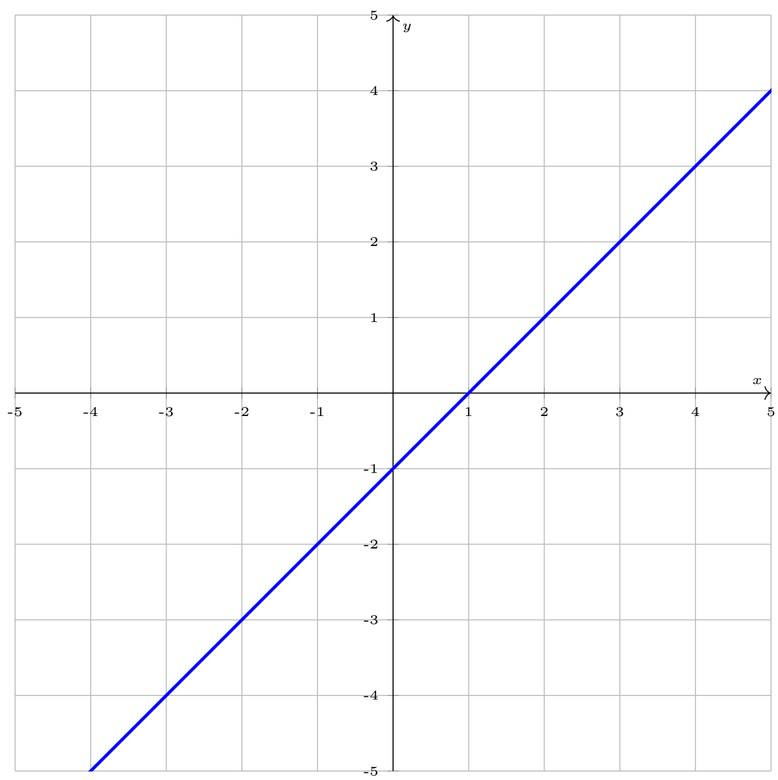
$$x - y \leq 1$$

Solution

We begin by graphing the line $x - y = 1$ using the intercepts:

$$(1, 0) \text{ and } (0, -1)$$

Since the inequality *includes* 1 (it is less than *or equal to*), we draw the line solid.

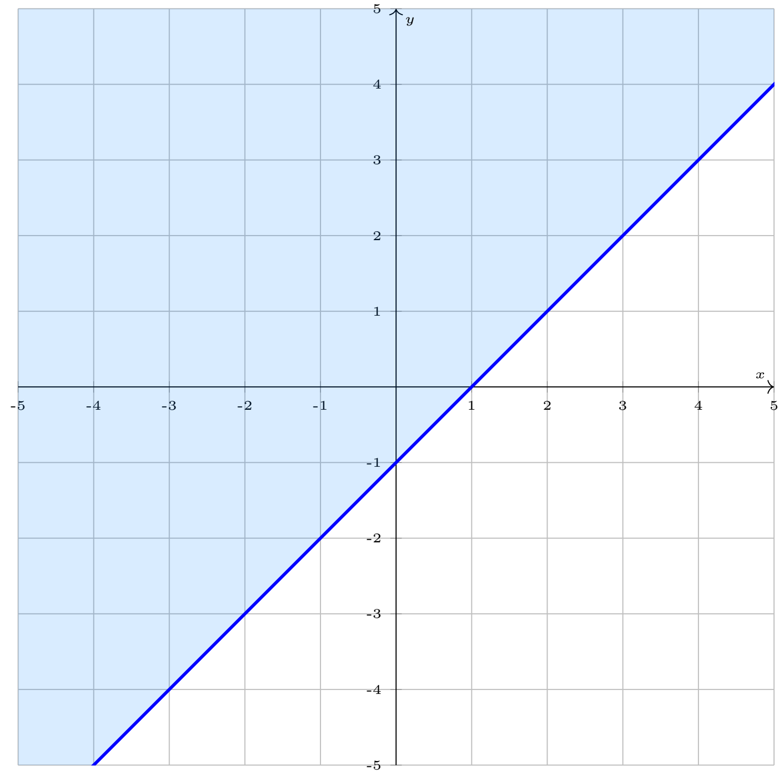


Now, to figure out which side to shade, we pick a test point. The simplest is the origin: $(0, 0)$. This is clearly above the line–if it satisfies the inequality, we shade the upper side; if not, we shade the lower side.

Check $(0, 0)$ in the inequality:

$$x - y \leq 1$$
$$0 - 0 \leq 1$$

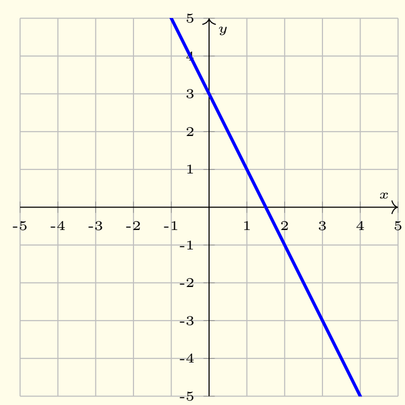
This is clearly true, so the side containing $(0,0)$ (the upper side) is the solution set.



TRY IT

Which side of the line below should be shaded if we draw the solution set for the following inequality?

$$2x + y \geq 3$$



We picked $(0,0)$ as the test point, and we'll continue to do that whenever possible, because it makes it simple to evaluate the inequality. The only way we wouldn't be able to use it as our test point would be if the line passed through the origin; in that case we'd simply pick another test point clearly to one side or the other of the line.

Notice that if we wrote the inequality similar to slope-intercept form, like $y \geq x - 1$ in the example above, we would see that we need to shade the upper side of the line, since those are the points whose y-coordinates are *greater* than those on the line.

GRAPHING A LINEAR INEQUALITY EXAMPLE 2

Graph the solution set for the following inequality.

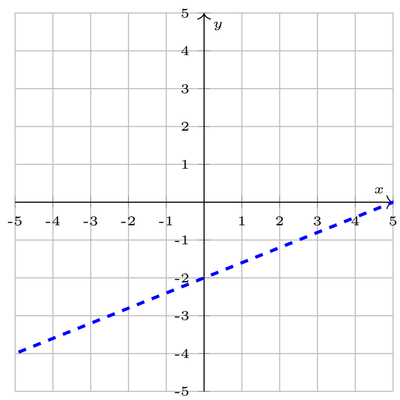
$$2x - 5y > 10$$

Solution

First, graph the line $2x - 5y = 10$. Again, we will use the intercepts to do this:

$$\text{The intercepts are } (5, 0) \text{ and } (0, -2)$$

Since the inequality does not include 10, we draw the line dashed.

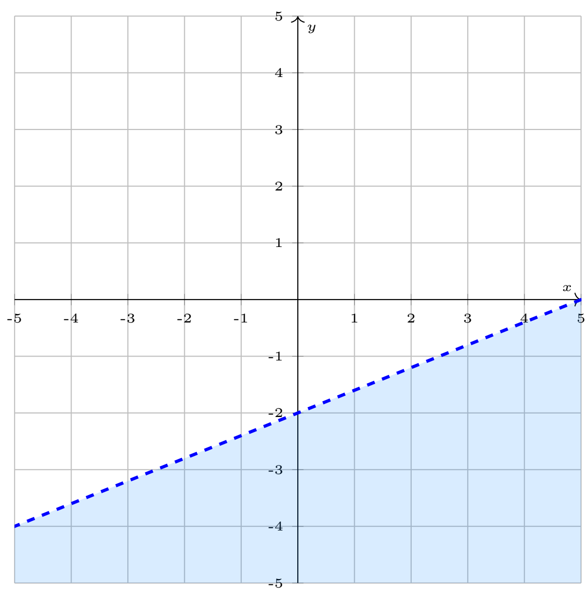


To figure out which side to shade, we again pick $(0, 0)$ as our test point.

Check $(0, 0)$ in the inequality:

$$2x - 5y > 10$$
$$2(0) - 5(0) > 10$$

This is false, so the side *not* containing $(0, 0)$ (the lower side) is the solution set.



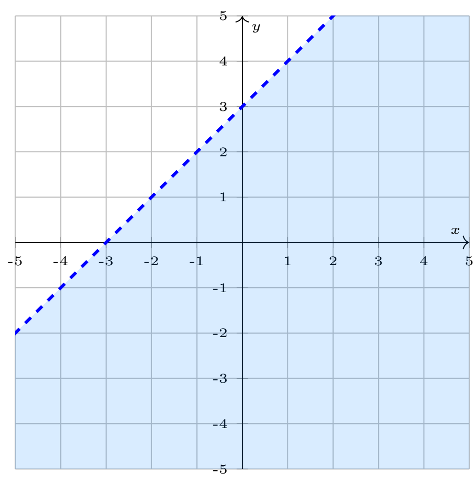
Systems of Inequalities

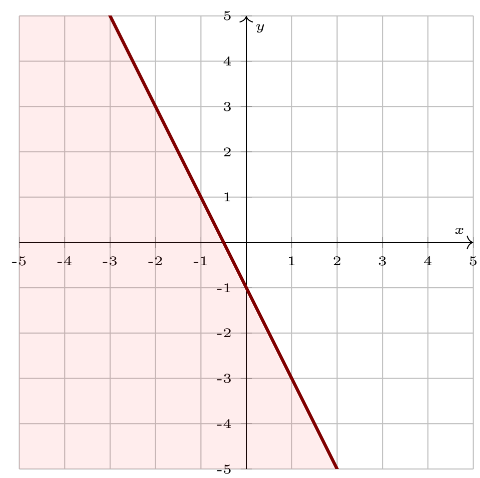
What about a *system* of inequalities? When it comes to the optimization problems, we'll have several inequalities, and we'll want to know what values satisfy *all* of them at once. Just like the solution to a system of equations is the point that lies on both lines–the place where the two lines overlap–the solution set for a system of inequalities is the *overlap* of the individual solution sets.

Solutions to a System of Linear Inequalities

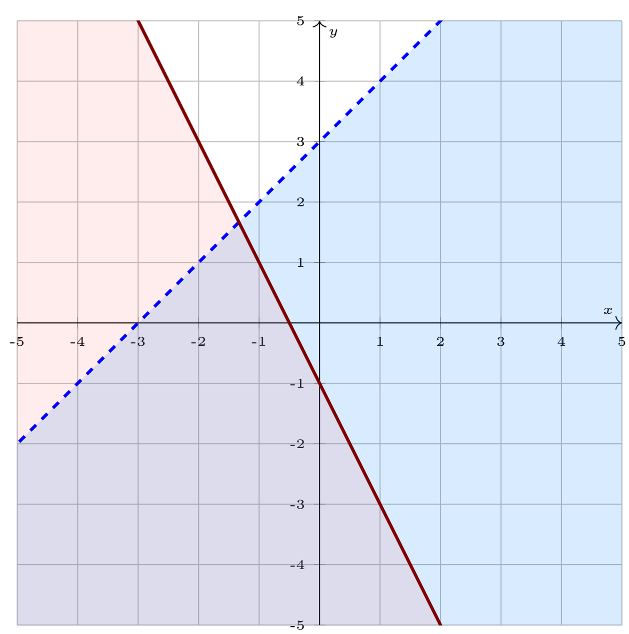
The solutions to a system of inequalities are the points that lie in the region where the solution sets of the individual inequalities overlap.

For instance, suppose we had two inequalities whose individual solution sets looked like the ones below.

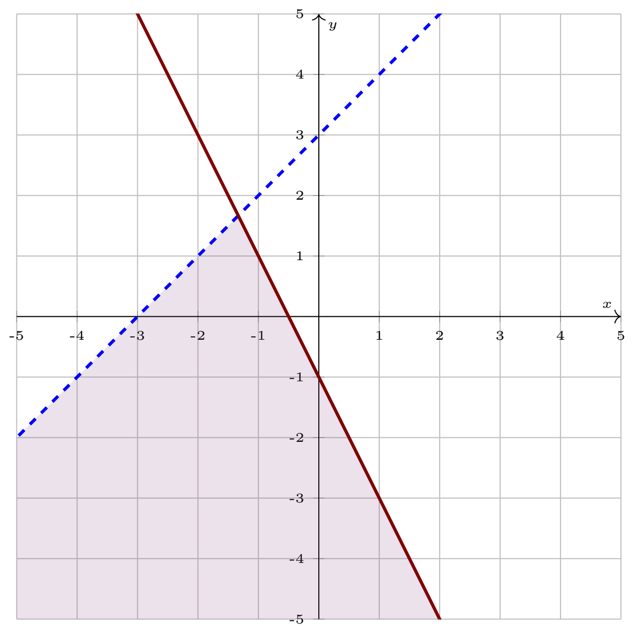




We could draw them on the same graph, and the combined solution set would be the overlapping region.



We can redraw the picture with only the overlapping region shaded, to make it clearer.



This illustrates how we graph the solution set for a system of linear inequalities: simply graph the solution set for each inequality and see where they overlap.

GRAPHING FOR A SYSTEM OF INEQUALITIES — EXAMPLE 3

Graph the solution set for the following system of inequalities.

$$2x + y < 4$$
$$x - y > 4$$

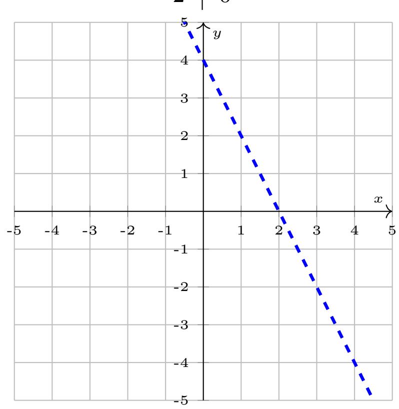
Solution

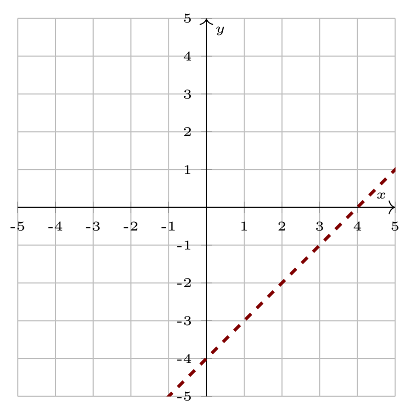
We begin by graphing the two corresponding lines, using the intercepts to graph. Note that both inequalities do *not* include equality, so we draw both lines dashed.

$2x + y = 4$ and $x - y = 4$

x	y
0	4
2	0

x	y
0	-4
4	0

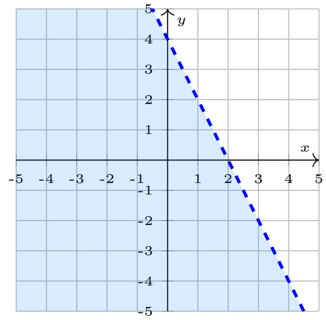


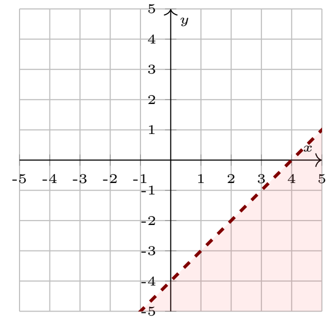


Next, test the origin on each inequality to see which side to shade:

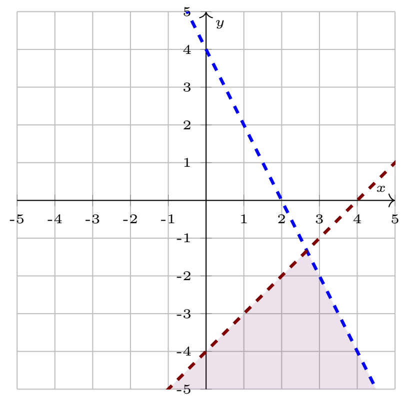
$2(0) + 0 < 4$ TRUE

$0 - 0 > 4$ FALSE





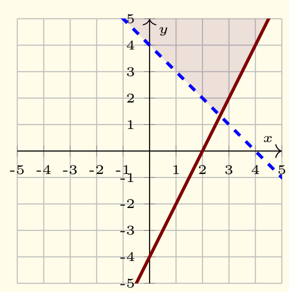
Finally, combine the pictures (we'll draw the final product with only the overlapping region shaded):



TRY IT

Is the graph below correct for the following system of inequalities? If not, what is wrong with it?

$$\begin{aligned} x + y &\leq 4 \\ y &> 2x - 4 \end{aligned}$$

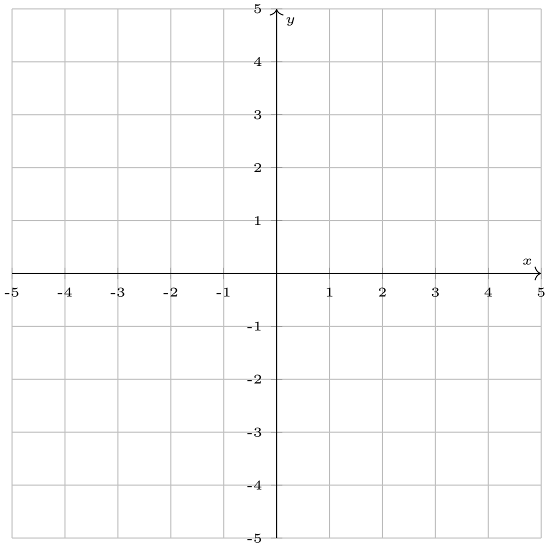


Exercises 5.3

In problems 1–4, graph the solution set for each linear inequality.

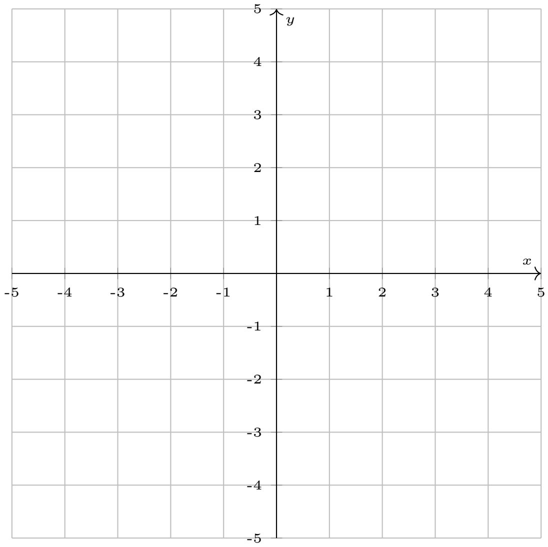
1.

$$3x + y < 4$$



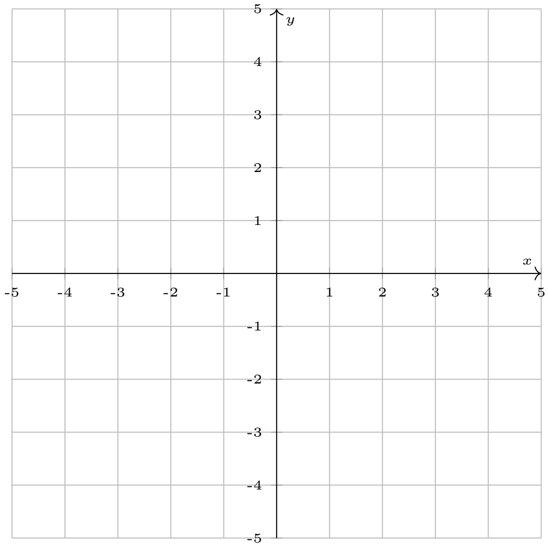
2.

$$x \geq y - 2$$



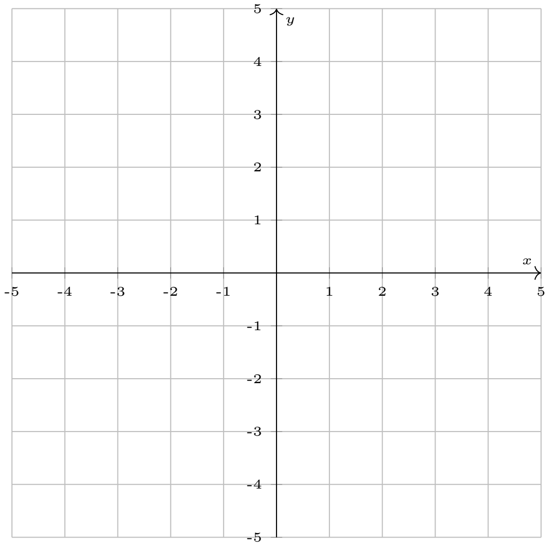
3.

$$2x - 2y > 4$$



4.

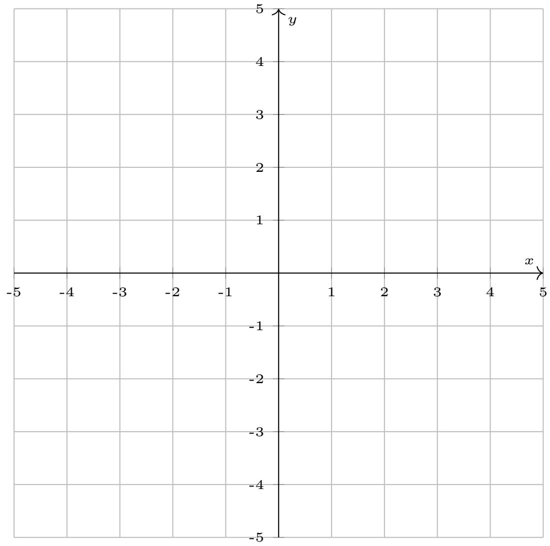
$$y \leq \frac{1}{2}x - 1$$



In problems 5–12, graph the solution set for each system of inequalities.

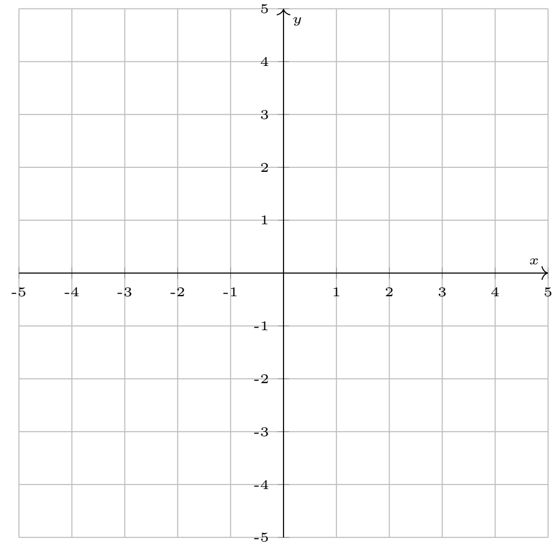
5.

$$x - 3y > -6$$
$$x \leq -3$$



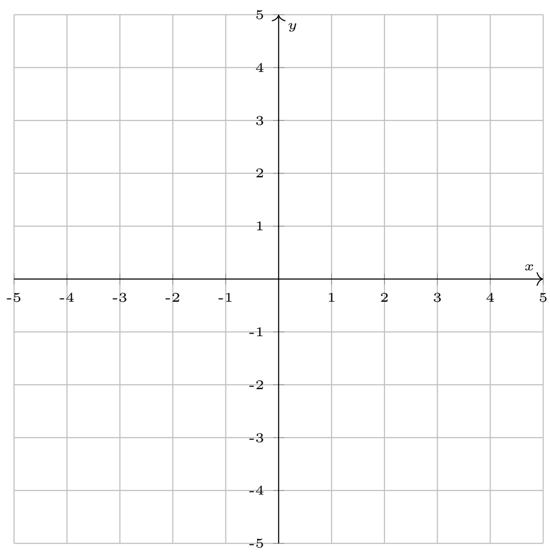
6.

$$3x + 4y < 12$$
$$y \geq -2x + 1$$



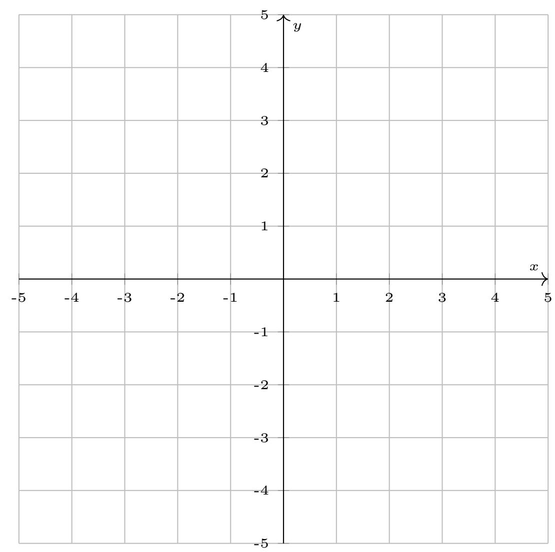
7.

$$x \geq y - 3$$
$$y > 2x + 1$$



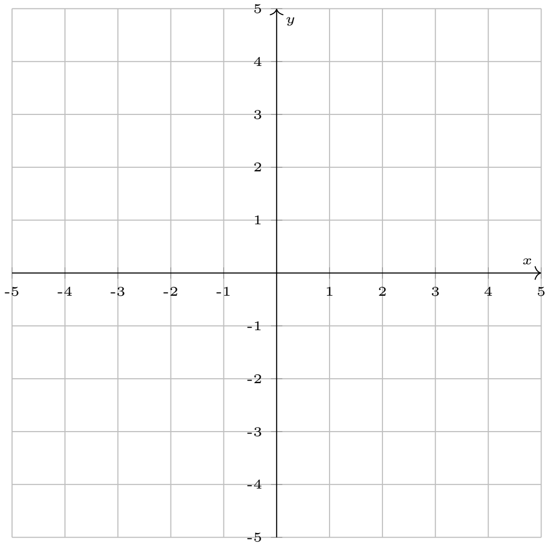
8.

$$y > -4$$
$$x \leq 2$$



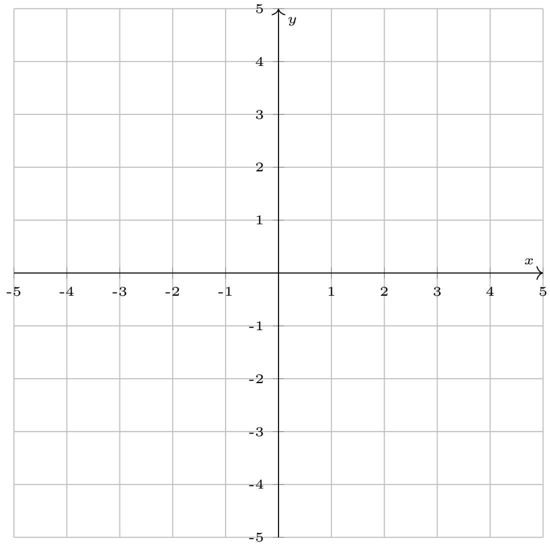
9.

$$x + y > -4$$
$$x - y \leq 0$$



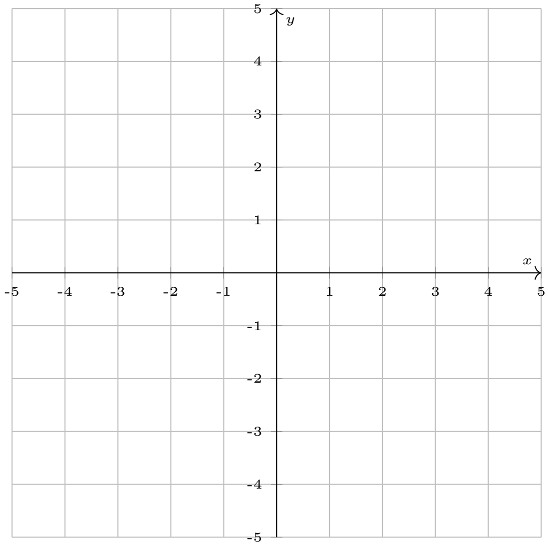
10.

$$2x - 5y \leq -10$$
$$y < 3$$



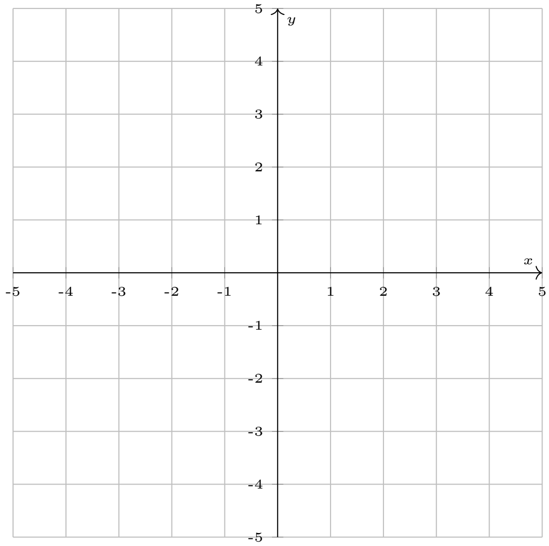
11.

$$4x + 5y < 20$$
$$y > -\frac{1}{3}x + 2$$



12.

$$y > \frac{3}{5}x - 4$$
$$y \leq -\frac{4}{3}x + 3$$



SECTION 5.4 Linear Programming

Since it began to be used for applications in the 1940s, linear programming has become a tremendous boon to businesses and governments all over the world, saving billions of dollars in the process. Nearly every industry–and nearly every major player within those industries–applies the ideas that we'll begin to examine in this section in order to find the best possible way to use their limited resources.

Linear Programming

Linear programming is a method used to find the maximum or minimum of some quantity like profit or cost, taking limitations, or *constraints*, into account.

We will look at relatively simple examples; real-world applications can involve thousands of equations and millions of possibilities to check, which explains why this application gained heavily in popularity once computers were available to solve such problems. The problems we will do will be solvable by hand, though–let's look at a simple example.

EXAMPLE 1

Suppose that Julie, one of Professor Yagodich's children, works as a dog walker and a babysitter for another family. She earns \$8 an hour walking the dog and \$9.75 an hour babysitting. She would like to earn as much as possible (this is what we want to maximize). However, she can work no more than 20 hours per week, the dog must be walked at least 5 hours per week, and the longest the family needs Julie to babysit each week is 7 hours (these are the *constraints*). Clearly, if there were no constraints, she could maximize her revenue by babysitting every hour of every day, but the limitations are what make this problem interesting (and realistic).

Step 1 **Identify the Variables** In order to express this situation mathematically, we will define variables that will allow us to write equations (and inequalities) to represent all of the information in the last paragraph. The *variables* are the quantities that can change, or more helpfully:

Variables

The variables are the quantities in the problem that it is our job to pick values for in order to get the optimal solution.

In other words, we want to maximize Julie's profit, and to do so, we have to decide how many hours she will spend walking the dogs and how many hours she will spend babysitting. These will be our variables.

The variables

$$x = \text{ hours spent each week dog walking}$$
$$y = \text{ hours spent each week babysitting}$$

In every problem we do, we will begin by defining the variables, and we will do so by asking what it is that we can control, what it is that we have to choose values for at the end. Also, by clearly labeling our variables (and thus not mixing up x and y), when we get to the end of the problem, we'll be less likely to make a mistake and answer the problem backwards.

Step 2 **Find the Objective Function** We've mentioned that the entire goal of linear programming is to maximize or minimize something, and we call this quantity that we want to optimize the **objective function**. In this case, the objective is for Julie to earn as much as she can, so we'll call the objective function r for revenue. When we write the objective function, it will depend on x and y, how many hours she spends at each job.

One simple way to make the process of defining the objective function easier is to split the function into the contribution from x and the contribution from y. In other words, part of Julie's profit will come from x, the hours she spends walking the dog, and part will come from y, the hours she spends babysitting.

$$r = \text{revenue from dog walking} + \text{revenue from babysitting}$$

The revenue from each job will be the number of hours spent doing that job (x or y) times the amount she is paid per hour for that job.

$$r = 8x + 9.75y$$

The objective function

Again, our goal will be to find the combination of x and y–within the allowed limitations–that will give the largest value for this objective function. However, we will set this function aside until the end of the problem; before we return to it, we must deal with the limitations that were stated in the problem.

Find the Constraints The constraints are the limitations on x and y that are listed in the problem. Looking back at the problem statement, we find the following three constraints:

Step 3

- Julie can work no more than 20 hours per week.
- The dog must be walked at least 5 hours per week.
- Julie cannot babysit more than 7 hours per week.

What we have to do now is to write these in terms of our variables. Just like with the objective function, it may be helpful to think of splitting these constraints into their contributions from x and y. We'll handle each constraint in order:

- The total number of hours she works is the sum of the number of hours she works at each job: $x + y$. This cannot be more than 20, so it must be *less than or equal to* 20:

$$x + y \leq 20$$

- She must spend at least (*greater than or equal to*) 5 hours walking the dog, so

$$x \geq 5$$

- She must spend less than or equal to 7 hours babysitting, so

$$y \leq 7$$

Notice the importance of keeping straight which variable represents which activity, so that we don't mix up x and y.

Summarizing the constraints:

$$\begin{aligned} x + y &\leq 20 \\ x &\geq 5 \\ y &\leq 7 \end{aligned}$$

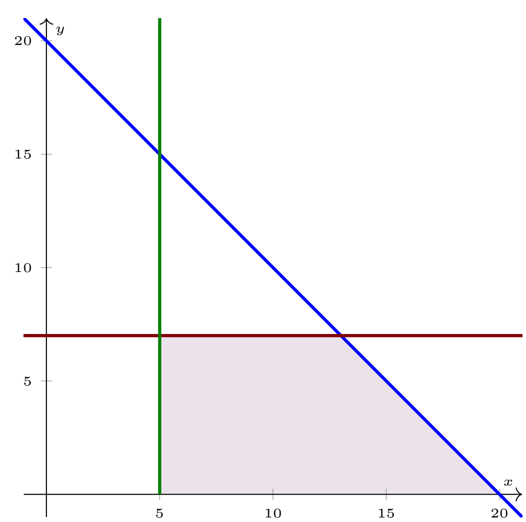
The constraints

Graph the Feasible Region Notice that the constraints form a system of linear inequalities, which we can graph.

Step 4

$$\begin{aligned} x + y &\leq 20 \\ x &\geq 5 \\ y &\leq 7 \end{aligned}$$

The feasible region



Note the *unstated* non-negative constraint

The shaded region is called the *feasible region* because it represents all the combinations (x, y) that are allowed by the constraints. Any point outside that region, if we substituted it into the system of inequalities, would violate one or more of the constraints. Notice that we used a constraint that we never stated: y must be positive, since Julie can't babysit fewer than 0 hours per week.

Feasible Region

The feasible region in a linear programming problem is the set of all points that satisfy all the constraints. The answer will be a point in the feasible region.

We now know that the optimal solution will be somewhere in that shaded region, but there are still too many possibilities to check each one individually. This is where the power of linear programming comes into play, with the following theorem.

Fundamental Theorem of Linear Programming (simple form)

The optimal point(s) will lie either at the corner of the feasible region or along one of its edges.

We'll come back a little bit later to think about why this makes sense, but for now we'll use it to answer the example we're working on. Now, instead of having to check *all* the points in the feasible region, we can simply check the four corners.

This is what makes linear programming so useful: we've turned an impossible problem into a simple one. All we have to do is find the coordinates of the corner points and evaluate the objective function for each of those combinations of x and y to find the best possible one.

Step 5

Find the Corner Points Finding the corner points is equivalent to solving four systems of equations; each time we solve a system of equations, we find the point where two lines cross.

1. Find where $x + y = 20$ and $y = 7$ intersect.

$$\begin{aligned} x + y &= 20 \\ y &= 7 \end{aligned}$$

   The second equation immediately tells us that the y-coordinate of the intersection is 7, and we can substitute that into the first equation to find that the x-coordinate must be 13.

2. Find where $x = 5$ and $y = 7$ intersect.

   For this one, we don't even have to try; we're immediately told the coordinates of the intersection.

3. Find where $x = 5$ and $y = 0$ intersect.

   Again, it is clear that this point is $(5, 0)$.

4. Find where $x + y = 20$ and $y = 0$ intersect.

   Knowing that $y = 0$, x must be 20.

Thus, the corner points–sometimes called *vertices*–are

The corner points (vertices)

$$(13, 7),\ (5, 7),\ (5, 0),\ \text{and}\ (20, 0)$$

These are the only four candidates for the optimal point; Julie will earn the most money by using one of these combinations of the number of hours spent doing each job. One of these combinations will also yield the *least* possible money she could earn while still doing these jobs, but that's not the point we really want to find.

Step 6

Evaluate the Objective Function at the Corners Since we know that one of these four points is the optimal point, all we have to do is find what Julie's revenue would be if she worked each combination of hours.

Remember that the objective function (from Step 2) is

$$r = 8x + 9.75y.$$

We evaluate this function at the four corner points, and we summarize our results in the table below.

x	y	$r = 8x + 9.75y$
13	7	172.25
5	7	108.25
5	0	40
20	0	160

The optimal solution

Conclusion

The optimal point is the row highlighted above: by walking the dog for 13 hours per week and babysitting for 7 hours per week, Julie will maximize her earnings.

This example illustrates the general process for solving linear programming problems: it all boils down to finding the few candidates for the optimal point (namely, the corners of the feasible region) and checking the objective function at each of these candidate points.

Solving Linear Programming Problems

1. Identify the variables. Look for the quantities in the problem for which you are asked to decide the value.
2. Find the objective function. Define the quantity you are asked to maximize or minimize in terms of the variables.
3. Find the constraints. Look in the problem statement for limitations, and then describe these in terms of the variables. Note that in most problems, it'll be implied that the variables are nonnegative, but rarely stated. Make sure to account for this.
4. Graph the feasible region. Write the constraints as a system of linear inequalities, then graph the solution set for the system (the overlap of the individual solutions).
5. Find the corner points. Solve as many systems of equations that you need to in order to find where all the constraint lines intersect.
6. Evaluate the objective function at the corners. Plug the x- and y-coordinates of each of the corner points into the objective function. The largest result will be the overall maximum, and the smallest result will be the overall minimum.

We'll illustrate two more examples in this section.

EXAMPLE 2

> Express Bike Shop offers custom bike kits. The standard kit requires 15 hours of shop time, 8 hours of painting time, and 1 hour of inspection time. The deluxe kit requires 12 hours of shop time, 12 hours of painting time, and 2 hours of inspection time. Including all the employees, the bike shop has 120 hours available for shop time, 72 hours of painting time, and 11 hours of inspection time available each week. How many customizations of each type should Express Bike Shop perform each week if each standard kit results in a profit of $175 and the deluxe kits each result in a profit of $275? What is the maximum profit?

Step 1 **Identify the Variables** Here we are asked how many customizations of each type to do, so these will be our variables.

$$x = \text{ number of standard kits}$$
$$y = \text{ number of deluxe kits}$$

Step 2 **Find the Objective Function** The goal here is to maximize profit, so we need to define the profit function in terms of how many of each kit is sold. The total profit is the profit that comes from each kind of kit, which is the number of kits times the profit per kit:

$$p = 175x + 275y$$

Step 3 **Find the Constraints** The limitations are the following:

- There are 120 hours of shop time, so we can use up to and including 120 hours between the two types of customizations.
- There are 72 hours of painting time.
- There are 11 hours of inspection time.
- The number of kits must be 0 or greater. This was never said, but it is clear that it must be true.

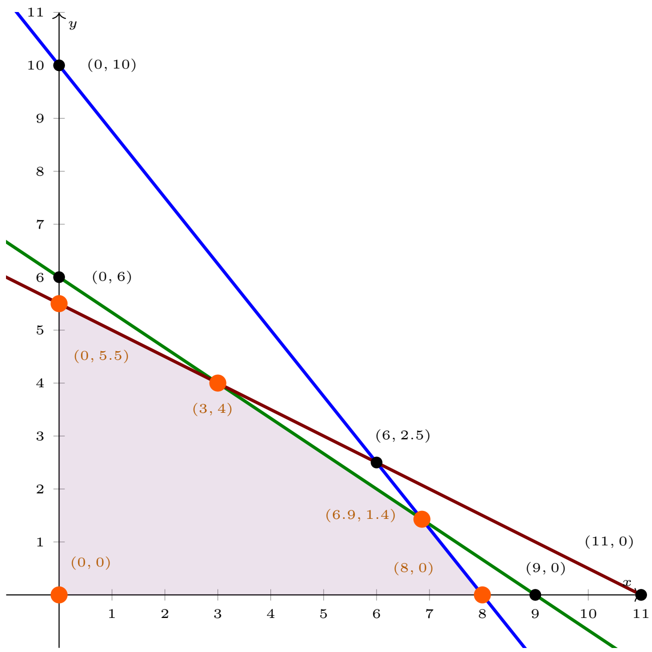
Writing the constraints in terms of the variables:

- The number of hours of shop time required is the sum of the number of hours required for each job times the number of jobs of that type. This must be less than or equal to 120.
$$15x + 12y \leq 120$$
- Similarly, for painting time:
$$8x + 12y \leq 72$$
- Finally, for inspection time:
$$1x + 2y \leq 11$$
- The implied nonnegative constraints:
$$x \geq 0$$
$$y \geq 0$$

Summarizing the constraints:

$$15x + 12y \leq 120$$
$$8x + 12y \leq 72$$
$$x + 2y \leq 11$$
$$x \geq 0$$
$$y \geq 0$$

Graph the Feasible Region The graph is shown below. Notice that the nonnegative constraints simply mean that we'll be limited to the upper-right quadrant of the coordinate plane. We graphed the lines using the intercepts, because then we have some of the corner points already labeled, saving us some time later. **Step 4**



Find the Corner Points By graphing using the intercepts, we found three of the corners along the way: **Step 5**

$$(0,0),\ (0,5.5),\ \text{and}\ (8,0)$$

To find the other two (already shown on the graph above, along with all the other intersections that we don't need), we need to solve the following systems of equations:

$$\begin{aligned} 8x + 12y &= 72 \\ x + 2y &= 11 \end{aligned} \qquad\qquad \begin{aligned} 15x + 12y &= 120 \\ 8x + 12y &= 72 \end{aligned}$$

Solution: $(3,4)$ Solution: $\left(\frac{48}{7}, \frac{10}{7}\right)$

We don't show the process of solving these systems of equations, but they can be done using either substitution or elimination (or graphing with a calculator).

Thus, the five vertices are

$$(0,0),\ (0,5.5),\ (3,4),\ \left(\frac{48}{7}, \frac{10}{7}\right),\ \text{and}\ (8,0)$$

Evaluate the Objective Function at the Corners If we evaluate the objective function from Step 2 at each of these vertices, we get the results summarized in the table below. **Step 6**

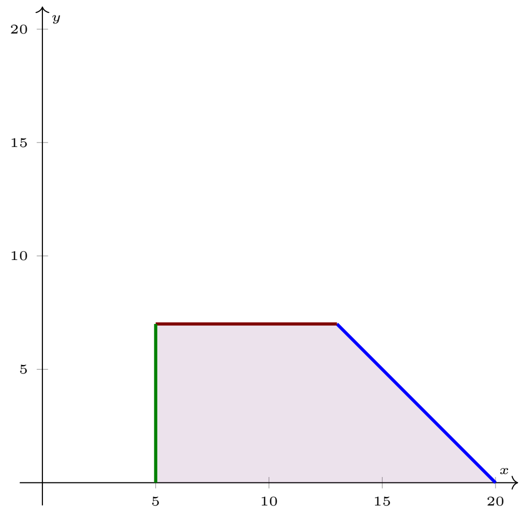
x	y	$p = 175x + 275y$
0	0	0
0	5.5	1512.5
3	4	1625
48/7	10/7	1592.86
8	0	1400

The optimal point is the row highlighted above: by selling 3 standard kits and 4 deluxe kits per week, the bike shop will maximize their profits. This maximum profit is $1625 per week. **Conclusion**

Why the Fundamental Theorem Works

We've used the theorem for each of the last two examples, accepting that the optimal value occurs at one of the corners (or along an edge, but neither example had that occur). But now we'd like to see why the theorem is true. We won't give a detailed proof, but we can make a convincing graphical argument.

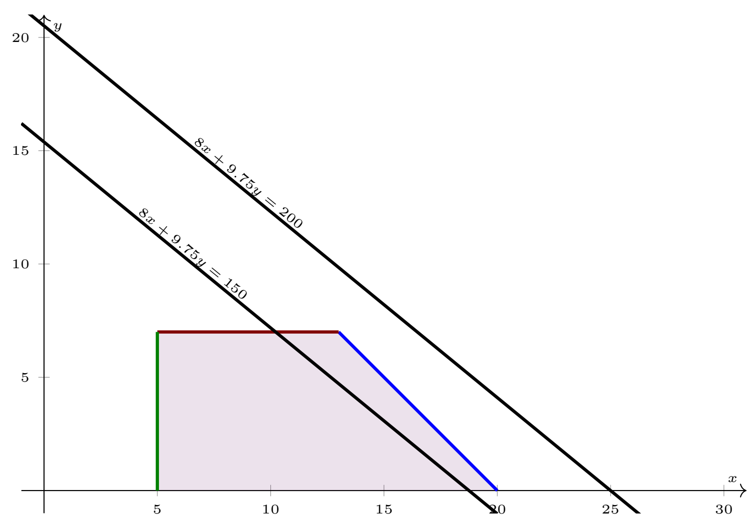
Suppose we have the feasible region from the first example.



Recall that the objective function in that example was $r = 8x + 9.75y$. Now we ask: could Julie make \$200? In that case, $200 = 8x + 9.75y$, which is a linear equation. If we graph it, it will show all the combinations of x and y (the number of hours spent at each job) for which she makes \$200.

However, notice that these points all lie outside the feasible region, which means that Julie cannot make \$200 and still satisfy all the constraints.

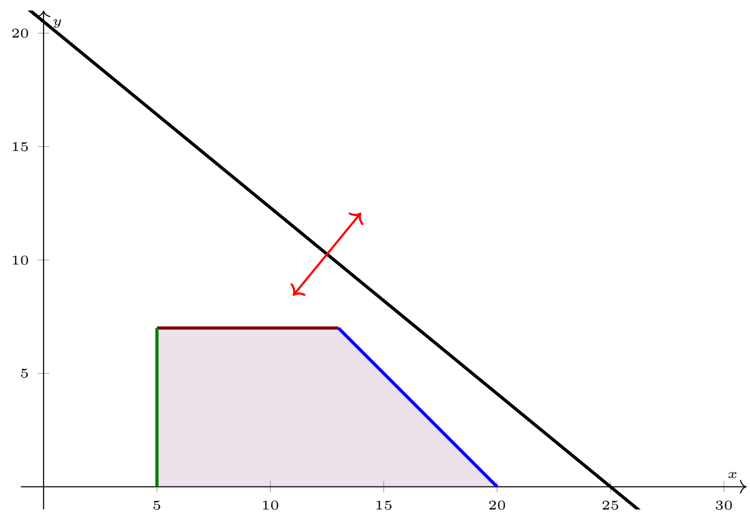
Okay, but what about \$150? Now the objective function becomes $150 = 8x + 9.75$, and we can graph that as well.



Immediately, we notice two things:

1. The second line cuts through the feasible region, meaning that there *are* cases where Julie can make \$150 and satisfy all the constraints.

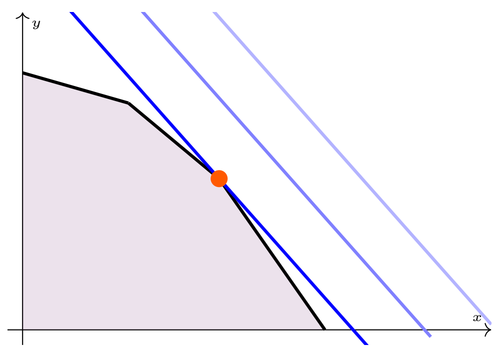
2. The two lines are parallel. If we drew a third line with a third value for Julie's revenue, it would also be parallel to these two, because the coefficients of x and y, which determine the slope, would not change. These parallel lines are called *level curves*, and changing the value of the revenue simply slides these level curves in and out.



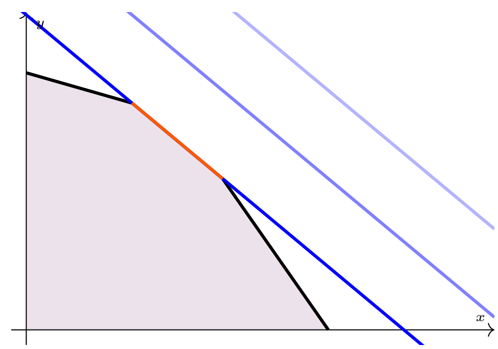
Changing the value of the objective function slides the level curve in and out

So it's impossible to make \$200, given the constraints, but it's possible to make \$150. However, notice that we could slide the \$150 line out a little bit before getting out of the feasible region, so \$150 is not the *best* that we can do.

In general, if we start with a value for the objective function that is too high to be possible, as we start to lower it we find that it will first hit the feasible region either at a corner or along an edge.



Optimal point at a corner



Optimal points along an edge

We haven't done any examples where the optimal points lie along an edge, but the principle remains the same. If two of the corner points have the same value for the objective function, then the optimal points are all the points on the edge that connects those two corners. In that case, the choice of any point along that edge will be an acceptable position to be in.

EXAMPLE 3

Here is a *very* simplified version of the problem facing the planners of the Berlin Airlift (in reality there were over fifty variables to consider, rather than just two).

The goal is to maximize the weekly cargo capacity of American and British planes flying into Berlin. The cargo capacity of an American plane is 30,000 cubic feet and the cargo capacity of a British plane is 20,000 cubic feet. However, there are only enough runways to allow 56 flights per week, plus the total cost per week is limited to \$300,000 (an American flight costs \$9000 and a British flight costs \$5000). Finally, only 512 crew members are available; each American flight requires 16 crew members and each British flight requires 8. How many American and British planes should be used to maximize the cargo capacity?

Step 1 **Identify the Variables** Our job is to decide how many of each type of plane to use.

$x =$ number of American planes
$y =$ number of British planes

Step 2 **Find the Objective Function** The goal here is to maximize cargo, so we look at how much cargo each type of plane can carry.

$$c = 30,000x + 20,000y$$

Step 3 **Find the Constraints** The limitations are the following:

- Only 56 flights total are allowed.

$$x + y \leq 56$$

- The total cost cannot exceed \$300,000.

$$9000x + 5000y \leq 300,000$$

- Only 512 crew members are available.

$$16x + 8y \leq 512$$

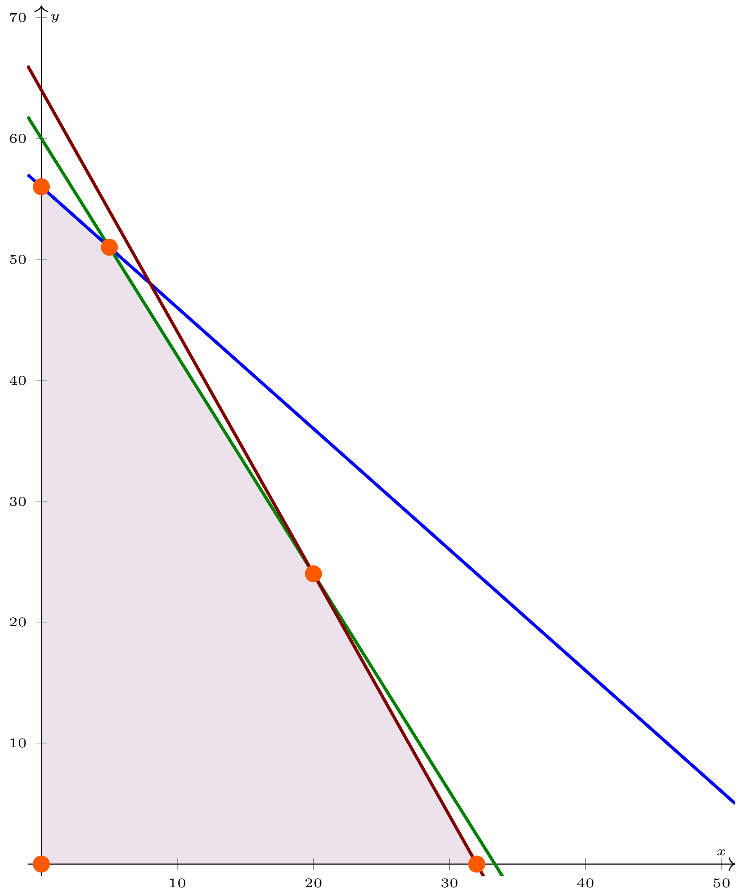
- As usual, it isn't stated, but it only makes sense if x and y are both nonnegative.

$$x \geq 0$$
$$y \geq 0$$

Summarizing the constraints:

$$x + y \leq 56$$
$$9000x + 5000y \leq 300,000$$
$$16x + 8y \leq 512$$
$$x \geq 0$$
$$y \geq 0$$

Graph the Feasible Region The graph is shown below. **Step 4**



Find the Corner Points By graphing using the intercepts, we found three of the corners along the way: **Step 5**

$$(0,0),\ (0,56),\ \text{and}\ (32,0)$$

To find the other two, we need to solve the following systems of equations:

$$9000x + 5000y = 300,000$$
$$16x + 8y = 512$$

Solution: $(20, 24)$

$$x + y = 56$$
$$9000x + 5000y = 300,000$$

Solution: $(5, 51)$

Again, for sake of space, we don't show the process of solving these systems of equations, but they can be done using either substitution or elimination (or graphing with a calculator).

Thus, the five vertices are

$$(0,0),\ (0,56),\ (5,51),\ (20,24),\ \text{and}\ (32,0)$$

Evaluate the Objective Function at the Corners If we evaluate the objective function from Step 2 at each of these vertices, we get the results summarized in the table below. **Step 6**

x	y	$c = 30,000x + 20,000y$
0	0	0
0	56	1,120,000
5	51	1,170,000
20	24	1,080,000
32	0	960,000

The optimal point is the row highlighted above: by using 5 American planes and 51 British planes, the Allies will maximize the cargo flying into Berlin. The maximum cargo that can be carried under these conditions is 1,170,000 cubic feet per week. **Conclusion**

Exercises 5.4

In problems 1–4, write an objective function for the given situation in terms of the variables defined.

1. A fence company sells wooden and metal fences. Let x represent the number of wooden fences they sell and let y represent the number of metal fences. They make a profit of \$320 for each wooden fence and \$280 for each metal fence.

2. You are placing mulch in your yard, and you find that pine chips cost \$2 per bag, while oak chips cost \$4 per bag. You want to minimize total cost. Let x be the number of bags of pine chips and y be the number of bags of oak chips.

3. An auto repair shop offers tire rotations and oil changes. They make a profit of \$25 on each oil change and a profit of \$18 on each tire rotation. Let x be the number of oil changes and y be the number of tire rotations.

4. Taking a pill of Medicine A gives you 6 mg of an undesired substance, and Medicine B gives you 8 mg of the undesired substance (you want to minimize the amount of this substance). Let x be the number of A pills you take and y be the number of B pills you take.

In problems 5–8, write an inequality to represent each constraint given.

5. Manufacturing one chair (x) requires 6 ft of aluminum tube, and manufacturing one table (y) requires 12 ft of aluminum tube. There are 500 ft of aluminum tube available.

6. Each oil change (x) takes 20 minutes and each tire rotation (y) takes 15 minutes. There are a total of 2400 minutes available.

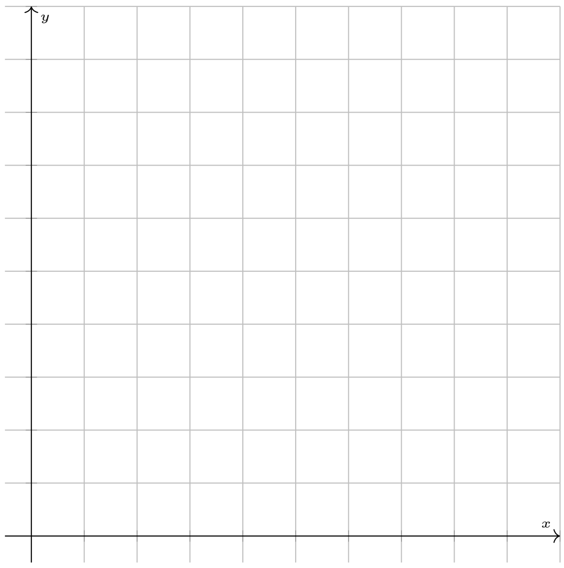
7. You must take at least 3 pills of Medicine A (x) and at least 2 pills of Medicine B (y).

8. Mowing a large yard (x) uses 0.25 gallons of gasoline, and mowing a small yard (y) uses 0.1 gallons of gasoline. There are 5 gallons of gasoline available.

In problems 9–12, graph each feasible region and list the corner points.

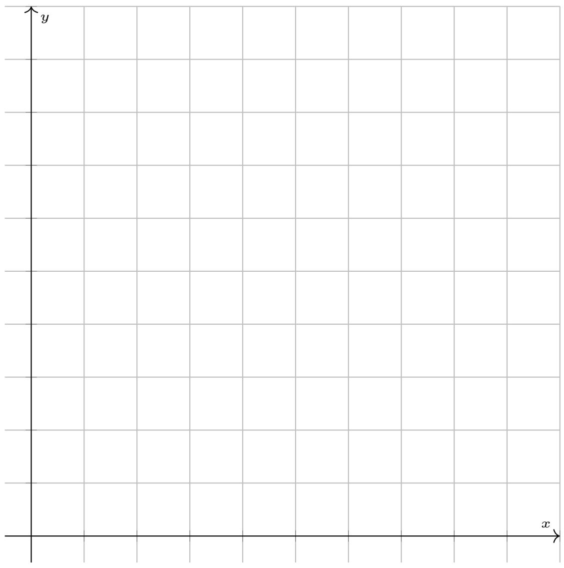
9. Feasible region:

$$\begin{aligned} 2x + 4y &\leq 20 \\ 4x + 2y &\leq 16 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$



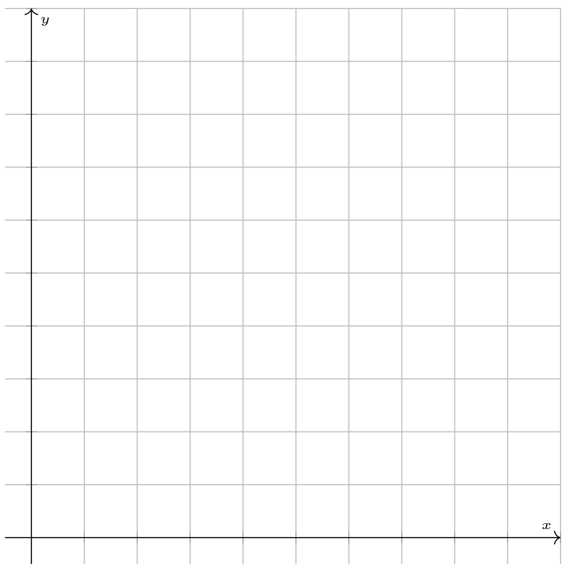
10. Feasible region:

$$\begin{aligned} 20x + 40y &\leq 160 \\ 18x + 9y &\leq 90 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$



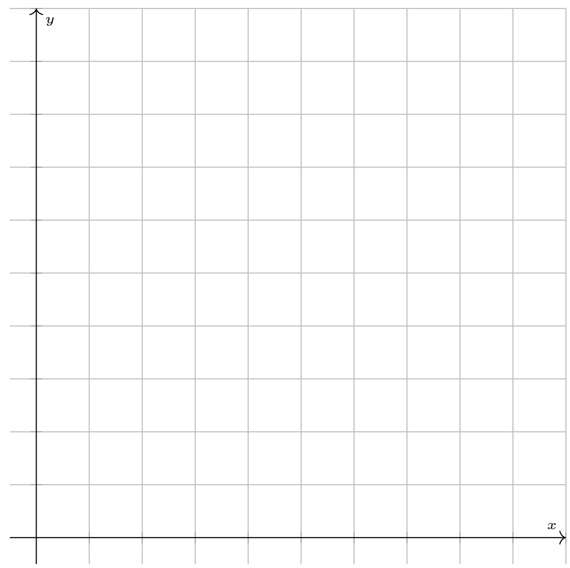
11. Feasible region:

$$\begin{aligned} 2x + y &\le 8 \\ x + 3y &\le 6 \\ x &\ge 0 \\ y &\ge 0 \end{aligned}$$



12. Feasible region:

$$\begin{aligned} 15x + 5y &\le 75 \\ 9x + 9y &\le 81 \\ x &\ge 0 \\ y &\ge 0 \end{aligned}$$



In problems 13–16, pick which of the given corner points maximizes the given objective function.

13. Objective function:

$$p = 5x + 7y$$

Corners:

$$(0,0),\ (8,4),\ (6,5),\ (0,8),\ \text{and } (12,0)$$

14. Objective function:

$$p = 20x + 12y$$

Corners:

$$(18,9),\ (20,0),\ (0,36),\ \text{and } (12,10)$$

15. Objective function:

$$p = 95x + 72y$$

Corners:

$$(0,0),\ (5,7),\ (3,9),\ (0,10),\ \text{and } (7,0)$$

16. Objective function:

$$p = 28x + 32y$$

Corners:

$$(0,0),\ (12,9),\ (9,15),\ (0,13),\ \text{and } (15,0)$$

17. A graphic designer can design a magazine cover or a logo. Her company makes a profit of \$800 for each magazine cover and \$500 for each logo. She estimates that it takes her 4 hours of brainstorming for a magazine cover and 2 hours of brainstorming for a logo. She'd like to keep the total brainstorming time under 24 hours a week. Further, she estimates that it takes her 2 hours to lay out a magazine cover and 0.5 hours to sketch up a logo, and she must fit this into 10 hours a week. Her boss requires her to design no more than 4 logos for each magazine cover she designs. How many of each should she design in order to maximize the company's profits? What is the maximum profit?

18. A manufacturer of ski clothing makes ski pants and ski jackets. The profit on a pair of ski pants is \$2.00 and the profit on a jacket is \$1.50. Both pants and jackets require the work of sewing operators and cutters. There are 60 minutes of sewing operator time and 48 minutes of cutter time available. It takes 8 minutes to sew one pair of ski pants and 4 minutes to sew one jacket. Cutters take 4 minutes on pants and 8 minutes on a jacket. Find the maximum profit and the number of pants and jackets the manufacturer should make in order to maximize the profit.

19. An automotive plant makes the Quartz and the Pacer. The plant has a maximum production capacity of 1200 cars per week, and they can make at most 600 Quartz cars and 800 Pacers each week. If the profit on a Quartz is \$500 and the profit on a Pacer is \$800, find how many of each type of car the plant should produce. What is the maximum profit?

20. A farmer has a field of 70 acres in which he plants potatoes and corn. The seed for potatoes costs \$20/acre, the seed for corn costs \$60/acre, and the farmer has set aside \$3000 to spend on seed. The profit per acre of potatoes is \$150 and the profit for corn is \$50 an acre. How many acres of each should the farmer plant? What is the maximum profit?

21. A manufacturer produces two models of mountain bikes. The times (in hours) required for assembling and painting each model are given by the following table:

	Model A	Model B
Assembling	5	4
Painting	2	3

The maximum total weekly hours available in the assembly department and the painting department are 200 hours and 108 hours, respectively. The profits per unit are \$25 for Model A and \$15 for Model B. How many of each type should be produced to maximize profit? What is the maximum profit?

22. A student earns \$10 per hour for tutoring and \$7 per hour as a teacher's aide. To have enough free time for studies, he can work no more than 20 hours per week. The tutoring center requires that each tutor spends at least three hours per week tutoring, but no more than eight hours per week. How many hours should he work to maximize his earnings? What are the maximum earnings?