Versatile Mathematics

COMMON MATHEMATICAL APPLICATIONS

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Attributions This book benefited tremendously from others who went before and freely shared their creative work. The following is a short list of those whom we have to thank for their work and their generosity in contributing to the free and open sharing of knowledge.

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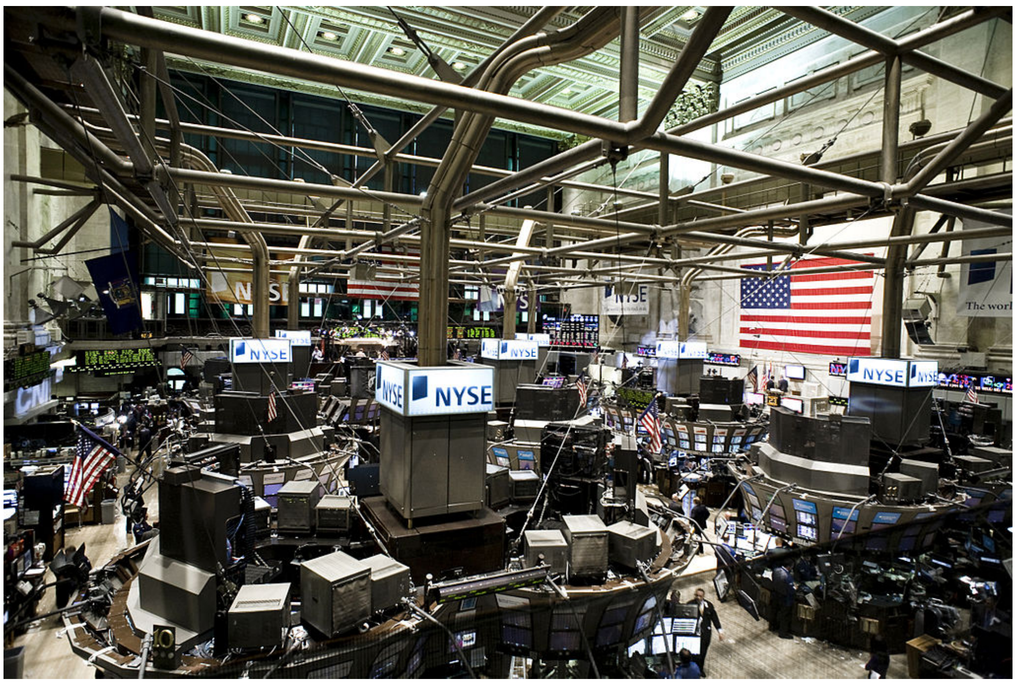

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The Great Recession, as it came to be called, began in 2007 when a housing bubble in the United States burst, leading to an economic decline that spread across the globe and lasted until 2009, with effects of the recession lasting even longer. Although the causes of the recession were complicated, the root of the problem was the subprime mortgage market; thousands of people were offered mortgages to buy homes beyond what they could afford, and the true cost of the mortgages were obscured. If they had truly known what they were getting themselves into, perhaps many of these hapless borrowers could have avoided defaulting on their mortgages and losing their homes.

The purpose of this chapter is to train you to be a careful, knowledgeable consumer. No other area in this book will be as immediately and broadly applicable as this material on financial mathematics. Here you'll begin to apply mathematical techniques to everyday financial management. How much should you budget for a new car? When should you start saving for retirement? How is your federal income tax calculated? These questions and ones like them will find answers in this chapter, as we investigate everything from sales taxes to credit cards. By understanding how to manage your personal finances, you can protect yourself and take control of your financial future.

SECTION 1.1 Introduction

In the 1930s and 1940s, a young boy in Omaha, Nebraska spent nearly every spare hour in some entrepreneurial pursuit or another, ranging from delivering newspapers to purchasing and installing used pinball machines. This single-minded focus on business meant that by the time he finished college, he had saved the equivalent of over $100,000 in modern terms.

This boy, Warren Buffett, certainly benefited from some privileges, such as having the time to work and being able to keep everything he earned, but it is undeniable that his work ethic was impressive. He didn't stop with those savings, though; through long-term investment, he parlayed those funds into one of the largest fortunes in the world (at the time of writing, Warren Buffett is the fourth-richest person in the world). The "Oracle of Omaha," who still lives in the home that he bought in 1957 for $31,500, has pledged to give away 99% of his vast fortune to charity.

Why do we begin with this story? There are many other investors who were not so fortunate, but Warren Buffett is a dramatic example of the old adage, "It takes money to make money." One of the most notable things about Buffett is his frugality; rather than spending his money, he has spent decades re-investing it. Perhaps we could rephrase the adage in a slightly less pithy way:

Money is expected to grow over time.

Take a look, for instance, at a graph of the Dow Jones Industrial Average over the past 30 years. The DJIA is a value calculated by adjusting the sum of the prices of the stocks for thirty major companies such as Coca-Cola, Nike, Microsoft, and Walmart. It can be used to give a quick snapshot of how the stock market in general is behaving.

Source: Macrotrends.net

The value of the Dow Jones index is fairly volatile, and there are notable dips (notice the big drop in 2007, when the Great Recession began, and how the graph began to creep up again in 2009, as the recession eased). However, the overall trend is unmistakable: if you zoom out, the value of the stock market has increased steadily, and is expected to continue growing, even if there are short-term drops.

Time Value of Money

Let's take a smaller-scale example. Suppose you want to start a small business; let's say for the sake of argument that you'll be starting a landscaping company. You immediately run into a problem, though: if you had all the equipment you needed, you'd be able to start making money right away by offering your services. However, you don't have everything you need. You'll need a truck and a trailer, mowers and other equipment, and some form of advertising (business cards, signs, a website, etc.). What you've run into is that same principle: "It takes money to make money." If you had the money to start your business, you could invest in your landscaping company and use that to make more money.

So what do you do? The solution is to go to a bank and take out a small-business loan, because the bank has the cash on hand, and they recognize that if they loan it to you, you'll be able to—through your hard work—multiply that money several times over, so you'll be able to repay them.

This is the foundation of the principle that we expect money to grow over time. If you have some money today beyond what you need to cover living expenses, you can find some way to use it—either by adding your own hard work to it or finding somebody else who can do that—and by using it wisely, you can receive back more than you put in.

Time Value of Money

Money is expected to grow over time.

We'll have specific formulas for this later, but for now, all we need is the basic principle. For instance, \$100 today may be expected to be worth \$110 a year from now. In this example, the **present value** is \$100, and the **future value** is \$110.

"Money is of the prolific, generating nature. Money can beget money, and its offspring can beget more, and so on."
-Benjamin Franklin

Since this is the basis of the entire economy, we can simplify the big picture by envisioning money as something that grows. There are many implications of this, but the most significant one is this: there is value to holding money right now. Because of that, if you want to borrow money, you not only have to pay it back at some point, but you also have to pay extra on top of that for the privilege of holding onto the money for some time–this brings us to the crucial concept of **interest**.

Think back to the example of the landscaping company: you go to the bank and take out a small-business loan. When you do, they will lay out the terms of the loan. For instance, say you take out a loan of \$100,000 for 10 years. That means that by the end of the 10 years, you'll have to pay back all of the original \$100,000, plus whatever interest the bank adds on top. The question of **how** you pay it back, whether in one lump sum at the end or in smaller regular payments, will be addressed in later sections of this chapter.

Loans

Here are a few terms that will be used frequently throughout the chapter:

- **Principal**: The amount of money borrowed
- **Interest**: The fee added to the principal, which must also be paid
- **Life of the loan**: The amount of time until the loan must be paid off

In the example above, the principal is \$100,000 and the life of the loan is 10 years.

What about the interest?

Interest Rates

We need a fair way to decide how much interest to charge for a particular loan. Let's start with a simple example.

Pretend that you are in the bank's position: you have money to loan out, and you're accepting loan applications. Two people send in applications:

- One requests \$100,000 to buy a foreclosed house, fix it up, and resell it
- The other requests \$500,000 to buy 5 foreclosed houses, fix them all up, and resell them

How much do you charge them in interest?

Clearly it wouldn't make sense to charge a flat rate. Since both are making the same investment, but the second person is doing five times as much as the first, it's reasonable to assume that they will also see about five times the return as the first person.

Therefore, **interest scales with the amount of the loan**. This means that interest is always a fraction of the principal, although interest rates are described using percentages.

Interest Rates

Interest is always defined as a percentage of the principal.

For instance, if you take out a loan of \$100 for a year and agree to pay 5% interest, that means that at the end of the year you will owe \$100 plus 5% of \$100.

This means that we need to be comfortable working with percentages before we can solve the problems later in this chapter. Because of this, we'll include a short discussion of percentages here. More examples can be found in the review chapter included at the end of the book, and the next section will cover applied problems that use percentages.

Percentages

Percent: number of hundredths

A percentage is simply another way to represent a fraction or a decimal. The word "percent" means "per 100," or "number of hundredths."

Percents, Fractions, and Decimals

Since percent (%) means "number of hundredths," we can convert decimals to percents by multiplying by 100 (or moving the decimal point two places to the right).

We can convert fractions to percents the same way by first writing them as decimals.

EXAMPLE 1 CONVERTING TO PERCENTAGES

Convert each of the following to a percentage:

(a) $\frac{2}{5}$ (b) 0.15 (c) $\frac{9}{2}$

Solution

(a) $\frac{2}{5} = 0.40 = 40\%$ (b) $0.15 = 15\%$ (c) $\frac{9}{2} = 4.50 = 450\%$

TRY IT

Convert each of the following to a percentage:

(a) $\frac{3}{5}$ (b) 0.7 (c) 2

This process can be reversed to convert a percentage to a decimal, which we'll want to do frequently in later sections of this chapter.

To do so, simply remove the percent symbol and divide the percentage by 100 (i.e. move the decimal point two places to the left).

EXAMPLE 2 CONVERTING PERCENTS TO DECIMALS

Convert the following percentages to decimals:

(a) 28% (b) 104% (c) 0.37%

Solution

(a) $28\% = 0.28$ (b) $104\% = 1.04$ (c) $0.37\% = 0.0037$

What's Coming Next

With that, we're ready to dive into the rest of the chapter and explore all kinds of financial concepts. The most important principle that you need to carry with you is the simple idea we've seen several times: money is expected to grow over time, so whenever someone borrows money, the lender expects to not only get the principal back, but also some percentage in interest. The same concept applies when you put money into a bank account or other investment: you can expect to receive interest as payment for that account holding your money.

Here's what to expect in the rest of the chapter:

Section 1.2: Applied Percentage Problems Solve word problems that involve percentages, like "If a $20 shirt is discounted by 15%, what's the sale price?"

Section 1.3: Simple and Compound Interest Learn about compound interest and how it makes investments grow with incredible speed.

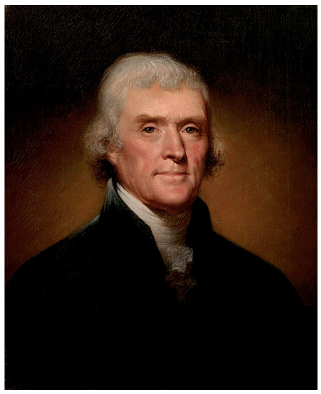
Section 1.4: Saving for Retirement It may seem far away now, but if you learn nothing else from this chapter, remember this: start saving for retirement **now**, even if you can only afford to save a little at a time.

Section 1.5: Mortgages and Credit Cards Car payments, home mortgages, and credit cards are all examples of *installment loans*, which are paid back with regular payments.

Section 1.6: Income Tax The endlessly quotable Benjamin Franklin famously said, "In this world nothing can be said to be certain, except death and taxes." Learn the basics of how income tax is calculated.

SECTION 1.2 Applied Percentage Problems

In 1804, Thomas Jefferson won the U.S. presidential election, edging out Charles C. Pinckney by 65,191 votes. A little over 100 years later, in the 1908 election, William Howard Taft beat William Jennings Bryan by 1,269,356 votes. The question is, who had a wider margin of victory?



Thomas Jefferson

William Howard Taft

Although Taft won his election by more votes than Jefferson, you've probably already spotted why this is misleading on its own: there were more votes cast in 1908 than in 1804. In order to compare these two values fairly, we need more information; specifically, we need to know how many votes were counted in total each year.

There were only 143,029 total votes in 1804, compared to 14,889,239 in 1908. We need a way to scale the margins of victory based on these totals; to do this, we can find what **percentage** of the total votes is represented by the margin.

Jefferson

$$\frac{65,191}{143,029} = 0.456 = \text{45.6\%}$$

Taft

$$\frac{1,269,356}{14,889,239} = 0.085 = \text{8.5\%}$$

By putting both numbers in context, we can tell that Jefferson won by a historic margin; the *difference* between him and his opponent was nearly half of all voters, and he won over 70% of all votes. Although Taft still won by a fairly large percentage, it was nowhere close to Jefferson's win.

Why Use Percentages?

Percentages are used to add context to a number, and to scale numbers so that they can be compared.

Percentages give us a consistent way to scale numbers. This is important because it's often hard for us to put numbers–especially large numbers–in context. For instance, a few years ago, an ad for an accounting firm claimed that Americans left a billion dollars behind by not getting the most out of their tax refunds, and they punctuated this with dramatic visuals of dropping pallets of cash.

For us as individuals, \$1,000,000,000 is certainly an eye-catching number, but that ad was trying to take advantage of how difficult it is to comprehend such large numbers. Once again, though, we can put this amount in context by asking the question, "What percentage is this of all taxes that were paid?"

In 2019, the U.S. federal government collected a total of \$3.5 trillion. If we divide \$1 billion by this total, we find out that the billion dollars represents less than 0.03% of all taxes collected, which is completely insignificant.

It turns out that there's an added benefit to calculating this percentage. Once we know that number, we can use it to estimate how much an individual "left behind," to borrow the terminology of the ad. If someone paid \$10,000 in federal taxes, for instance, we can calculate 0.03% of \$10,000. We'll discuss how to do this more later, but it turns out that this is a grand total of. . . around \$3.

Is that significant? Maybe, or maybe not, but remember that this was part of an ad designed to sell accounting services, and it seems unlikely they'd charge less than \$3 for their services.

Word Problems with Percentages

Consider the following three questions. You don't need to solve them yet; just look at their structure.

- What is 20% of 275?
- Eighteen is what percentage of 54?
- Twelve is 45.3% of what?

Do you notice any similarities? If you look carefully, you should see a common structure.

Applied Percentage Problems

Every applied percentage problem can be written in the following form:

$$A \text{ is } P \text{ percent of } B$$

Remember that when translating a sentence into its mathematical form, the word "is" gets replaced with the equals sign (since "is" really means "this thing and that thing are equal"), and the word "of" gets replaced with multiplication.
This means that every percentage problem is built around the equation

$$A = PB$$

and in every problem, two of these pieces are given. We simply have to solve for the missing piece.

If you'd like, you can use the equation above every time, substituting the appropriate values each time. The examples we started with would look like this, for instance:

- $A = (20\%)(275)$
- $18 = (P)(54)$
- $12 = (45.3\%)(B)$

However, in order to solve these problems in a more general way, we'll use x to represent the unknown piece each time. This means that for our examples, we would write

- $x = (20\%)(275)$
- $18 = (x)(54)$
- $12 = (45.3\%)(x)$

The reason for this is that once you get used to translating a word problem into its mathematical form and replacing the unknown with x, you can take that process and apply it to other kinds of problems.

Percentages or Decimals?

Remember that a percentage is simply another way to express a decimal or fraction (50% is the same as 0.5, which is also the same as 1/2).
It's important to remember that percentages are just meant for displaying numbers, not for doing calculations.

When doing calculations, always use the decimal form of a percentage.

As you'll see in many of the examples to follow, we'll give answers in both decimal form and percentage form, so you should be comfortable converting back and forth. It's common to give an answer in the same form as the problem is given, though, so if someone asks you a question about a percentage, you should give your answer as a percentage, even though you'll use decimals during the calculation phase.

EXAMPLE 1 SETTING UP PERCENTAGE QUESTIONS

Answer each of the following questions:

(a) What is 75% of 690?

(b) Forty is what percentage of 150?

(c) Eight is 62% of what?

Solution Just like we did at the start of the section, we'll start by rewriting each of these as an equation, with x in place of the word "what." Then we simply need to solve for x; be careful, though, to note whether x represents a percentage or not, so that we know how to phrase the answer.

(a) What is 75% of 690?

$$x = (75\%)(690)$$

This equation is already solved for us, in the sense that x is already alone on one side; all that we have to do is calculate 75% times 690, and we're done. However, we need to remember:

When doing calculations, always use the decimal form of a percentage.

That means that we need to rewrite 75% as a decimal; remember that to do this, we simply divide 75 by 100 (or drop the percentage symbol and move the decimal place two places to the left).

$$\begin{aligned} 75\% &= \frac{75}{100} = 0.75 \\ x &= (0.75)(690) \\ &= \boxed{517.5} \end{aligned}$$

(b) Forty is what percentage of 150?

$$40 = (x)(150)$$

To solve for x, we need to get rid of the 150 that is multiplied with it; we do this by dividing both sides by 150:

$$\begin{aligned} 40 &= (x)(150) \\ \frac{40}{150} &= x \\ 0.2667 &= x \end{aligned}$$

Notice that at the end of the calculation, we have the answer as a decimal, but since the question asked "what *percentage...*" we should give the answer as a percentage (multiply by 100 and add the percentage symbol):

$$x = \boxed{26.67\%}$$

(c) Eight is 62% of what?

$$8 = (62\%)(x)$$

Again, start by rewriting 62% as a decimal, then solve for x by dividing both sides of the equation by the result.

$$\begin{aligned} 8 &= (0.62)(x) \\ \frac{8}{0.62} &= x \\ \boxed{12.9} &= x \end{aligned}$$

TRY IT

Answer each of the following questions:

(a) What is 20% of 275?

(b) Eighteen is what percentage of 54?

(c) Twelve is 45.3% of what?

Here's the good news: if you can do those three examples, there's no problem in this section you can't solve. Every single word problem involving percentages follows the pattern of one of those three; all you have to do is figure out which pattern it is.

For instance, if you find that you paid \$4000 in federal income tax one year on a salary of \$45,662, you can figure out your *effective tax rate*, meaning what percentage of your income went to income tax. Just rephrase this in the standard form; what you really want to know is

\$4000 is what percentage of \$45,662?

Once you see it in that form, you can run through the same steps as the last example, and you'll soon see that to get the answer, you'll need to divide \$4000 by \$45,662.

It may be that you eventually get to the point where you can simply get used to jumping to that step and skipping the rest of the process. If you do, feel free to do so, but if you're not comfortable skipping steps, you can always deliberately set up the problem in this standard form.

EXAMPLE 2 COFFEE SURVEY

The Frederick News Post did a poll of 1500 people, asking them the following question: "Do you go to Starbucks at least 3 times a week?" Of those 1500 polled, 58% said yes. How many people replied yes to the survey?

Solution The goal is to rephrase this question in a familiar form, along the lines of

$$A \text{ is } P \text{ percent of } B$$

and figure out where the "what" goes (which part of the problem is unknown).

We know there are a total of 1500 people surveyed, and we know that **of** that total, 58% responded yes. This means that our question looks like

What is 58% of 1500?

If we replace "is" with the equals sign and "of" with multiplication, we get the following equation:

$$\begin{aligned} x &= (58\%)(1500) \\ &= (0.58)(1500) \\ &= \boxed{870} \end{aligned}$$

Thus, 870 people responded yes to the survey.

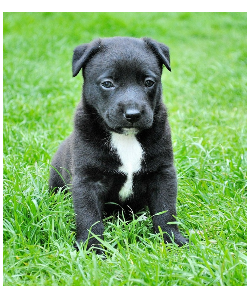
TRY IT If there are 6,233 students enrolled this semester, and 59% of those are women, how many women are attending the college this semester?

Let's try one with the shortcut: rather than setting up the full equation and solving for x, remember that if we're looking for a percentage, it's going to be one number divided by the other. Here's the trick: if you can't figure out which order to use for division, try one and see if your answer is on the right side of 100%.

In other words, say your question is "Five is what percentage of 10?" If you divide 10 by 5, you get $10/5 = 2$, which is equal to 200%. Since this is greater than 100%, this answer claims that 5 is greater than 10, so clearly that's the wrong order for division. If we go back and divide 5 by 10, we'll get the correct answer.

Of course, after you do this a few times, you'll get used to the idea that the number by itself on one side of the "is" gets divided by the number that's mentioned with the percentage.

EXAMPLE 3 DOG PEOPLE



In a survey of 400 people, 243 responded that they like dogs. What percentage of these people like dogs?

Let's try the shortcut method here. The question can be rephrased "What percentage of 400 is 243?" We know, since we're looking for the percentage, that we need to divide these two numbers, but the only question is what order to use for the division.

If we divide 400 by 243, we'll get a number larger than 1, or a percentage greater than 100%, which can't be right, so we need to divide 243 by 400:

$$= \frac{243}{400} = 0.6075 = \boxed{60.75\%}$$

Roughly 61% of people responded that they like dogs.

TRY IT Three of the nine sitting members of the U.S. Supreme Court are female. What percentage of the court is comprised of women?

Tips for Percentage Problems

In the standard problem

$$A \text{ is } P \text{ percent of } B$$

there are three possibilities for what to solve for: A, P, and B.

- To solve for the percentage P, you'll always need to divide one of the given numbers by the other. To find the order of division, think about whether the answer should be greater than or less than 1 (or 100%). If it should be greater than 1, divide the larger number by the smaller one, and vice versa.
  - **Note:** the number by itself on one side of the word "is" will be divided by the other
- If the percentage is given, and the goal is to find A or B, you'll need to either multiply or divide the given number by the percentage that's given. Which one you do depends on which number you're given:
  - If you know the number that's linked to the percentage by the word "of," multiply, since "of" is equivalent to multiplication.
  - Otherwise, divide the given number by the percentage.

SALES TAX — EXAMPLE 4

Suppose that you load a grocery cart with \$159 worth of groceries, and the local sales tax rate is 7%. How much tax do you pay, and what is the total cost of the groceries?

Solution

The sales tax rate tells you what percentage of the price will be added on top, so we'll calculate 7% of \$159 and add that to \$159:

$$(\$159)(7\%) = (\$159)(0.07) = \$11.13$$
$$+\,\$159 = \boxed{\$170.13}$$

Alternate Solution

We could simplify the solution by noticing that we start with 100% of the cost and add 7% to it, so we end up with 107% of the cost:

$$(\$159)(107\%) = (\$159)(1.07) = \$170.13$$

Taxes are not much fun to think about, so let's switch briefly to the happier side: discounts. When you go to a store that advertises a sale of 30% off, that means that whatever price is on the sticker will get slashed by 30%. Just like before, we can calculate the new price by finding 30% of the sticker price and subtracting that. Alternately, we could notice that if we're removing 30% of the price, we'll be left with 70% of the price. The next example illustrates a similar situation.

EXAMPLE 5 DISCOUNT

For months you have been wanting a 47” LCD flat screen television, but the price has been too high. The store is having a one-day sale on all televisions in the store. For one day only you can take 25% off any television. The regular price on the television you want is \$1099.

(a) What is the sale price?

(b) What will the final price be, including sales tax, if the sales tax rate is 8%?

(a) Since the sale takes 25% off the top of the price, the sale price will be 75% of the original price:

$$\begin{aligned}\text{Sale price} &= (75\%)(\$1099)\\ &= (0.75)(\$1099)\\ &= \boxed{\$824.25}\end{aligned}$$

(b) To add the sales tax, add 8% to this new price, so we can find 108% of \$824.25:

$$\begin{aligned}\text{Final price} &= (108\%)(\$824.25)\\ &= (1.08)(\$824.25)\\ &= \boxed{\$890.19}\end{aligned}$$

TRY IT

You have a 20% off coupon at Bed, Bath, and Beyond, and you’re ready to get some new towels. If you select some that are listed at \$30, and sales tax is 6%, how much will your final cost be at the register? Note that the discount is applied **before** the sales tax.

WATCH OUT! There’s a subtle pitfall lurking in the last example: you may think that since we’re removing 25% of the price for the sale, and adding 8% for the tax, we could simple combine everything into one step and simply remove 17% (the difference between 25% and 8%).

But that’s not right, and if you try that calculation, you’ll find that you get a different answer. What happened? It may be hard to spot at first, but notice that the sale is 25% *of the original price* and the tax is 8% *of the reduced price*. Since these percentages are tied to **different** bases, they can’t be combined.

This is an important idea, because this error is an easy one to see. In fact, once you understand this, you’ll likely spot this mistake in all sorts of places.

EXAMPLE 6 A TRICKY PERCENTAGE PROBLEM

Suppose you originally paid \$1200 in taxes. A year later taxes decreased by 20%, but the following year taxes increased by 20%. What do you pay in taxes at the end?

Solution

You may be tempted to jump to the conclusion that you’ll pay \$1200 in taxes at the end, since the 20% decrease was reversed by the 20% increase. Be careful, though: the decrease was 20% of *the original amount* and the increase was 20% of *the reduced amount*. Thus, the increase was *smaller* than the increase, so we expect to pay less than \$1200 at the end.

Specifically, after one year, the taxes would be

$$(\$1200)(0.80) = \$960.$$

After two years, the taxes would be

$$(\$960)(1.20) = \boxed{\$1152}$$

Percentage Increase or Decrease

There's another kind of problem involving percentages: we can talk about **percentage change**. For instance, you might hear a presidential candidate promise to cut taxes by 12%, or you may hear that there are 25% more hurricanes one year than the year before.

Of course, there's no such thing as a *new* percentage problem, because we know that every percentage problem has to fall into one of three categories, and we've done examples of all three. Our only job, then, is to find how to use what we already know to solve these problems that are stated a bit differently.

The most important thing to keep in mind with a percentage change problem is that

the change is given as a percentage **of the *original* amount**.

For instance, if you knew that there were originally 400 employees at a company, and 30 of them were laid off, this means that the size of the workforce decreased by whatever percentage 30 is of 400, so the problem boils down to "Thirty is what percentage of 400?" Having solved several problems of that type, we know to divide 30 by 400, and that leads to the following formula.

Percentage Change

The percentage change, or relative change, of a quantity is defined as the ratio of the total change to the original amount:

$$\text{Percentage Change} = \frac{\text{Total Change}}{\text{Original Amount}}$$

where the total change is the difference between the final and original amounts:

$$\text{Total Change} = \text{Final Amount} - \text{Original Amount}$$

If the percentage change is positive, the quantity increased; if it is negative, the quantity decreased.

As long as you don't forget to divide by the **original amount**, not the final amount, these problems are straightforward.

EXAMPLE 7 — CAR VALUE

The value of a car dropped from \$7400 to \$6800 over the last year. What percentage decrease is this?

First, find the absolute change:

$$\$7400 - \$6800 = \$600$$

Then divide this by the **original amount**:

$$\frac{\$600}{\$7400} = 0.081 = \boxed{8.1\%}$$

Thus, the value of the car dropped by 8.1%.

TRY IT

You go car shopping and find your dream car for \$11,000 sitting on the lot, but unfortunately you only have \$9,000 to pay for it. You offer the dealership the money you have and they accept your offer. What percentage did the dealership take off the car?

We can also turn this kind of problem around: if we know the percentage change, we can find either the original or final amount.

If we know the original amount and the percentage change, this is exactly like the problems with discounts and sales taxes from earlier; we have a starting value and a percentage adjustment, and need to find the final amount.

Going in the other direction is a bit trickier: to find the original amount when we know the final amount and the percentage change, we need to think carefully, because the change is given as a percentage of the *unknown* value.

EXAMPLE 8 AMOUNT BEFORE AN INCREASE

If the tuition and fees paid by the average student in the U.S. is \$9,139, and this is 17% more than the average five years ago, what was the average cost then?

The new amount—\$9,139—is 117% of the original cost. Since we don't know what the original cost was, we need a placeholder for it, so we'll call it x:

$$\$9,139 = (117\%)(x)$$

$$\begin{aligned} x &= \frac{\$9,139}{117\%} \\ &= \frac{\$9,139}{1.17} \\ &= \boxed{\$7,811} \end{aligned}$$

The original cost, then, was \$7,811 for the average student five years ago.

Relative change or percent differences are also useful when comparing quantities of different sizes. By calculating the percent difference, we can put everything on equal footing to make a meaningful comparison.

EXAMPLE 9 ENROLLMENT CHANGES

Generally, more students go to college every semester than the semester before. In the fall semester of 2008 there were 5,748 students enrolled at FCC; by the fall of 2009, the number had grown to 6,233. Compare these two numbers.

Solution

We could compare them by simply finding the difference, and saying that there were 485 more students in 2009 than in 2008, but out of context this number is mostly meaningless. At a small school like FCC, this may represent a dramatic change, but at a larger school like the author's alma mater of North Carolina State, 485 students wouldn't be as significant.

Instead, we can calculate this as a percentage change:

$$\frac{485}{5748} = 0.084 = 8.4\%$$

Enrollment rose by 8.4%, which is a modest but respectable gain.

TRY IT

If you took the SAT twice, scoring 500 on the verbal portion the first time and 620 the second time, by what percentage did your score increase?

This idea of putting different quantities on equal footing in order to compare them is extremely powerful, and crucial in many contexts. We'll do something similar later when we consider financial applications; there, we'll be interested in, for instance, comparing two loans with different interest rates and durations and deciding which is more advantageous. The details will differ, but the concept of finding common ground on which to compare different quantities comes up in many areas.

COMPARISONS — EXAMPLE 10

There are 435 Longhorn Steakhouse locations in the U.S. and 136 Ruth's Chris Steakhouse locations. Compare these two numbers.

Solution

Rather than simply saying that there are 299 more Longhorn locations that Ruth's Chris, let's use a meaningful measure like the percentage difference.

We could say that Longhorn is $\frac{299}{136} = 219.9\%$ larger, or that Ruth's Chris is $\frac{299}{435} = 68.7\%$ smaller. We could also say that Ruth's Chris is $\frac{136}{435} = 31.3\%$ of the size of Longhorn.

We've solved a wide array of problems using percentages, but every problem fit into one of the three categories introduced at the beginning of the section. In the next section, we'll apply these skills to calculate how interest accrues for different kinds of loans.

Exercises 1.2

For problems 1–8, convert each number to a percent.

1. $\frac{1}{2}$

2. 0.04

3. $\frac{3}{5}$

4. 0.79

5. 1.35

6. $\frac{10}{4}$

7. 12.5

8. 0.0378

For problems 9–16, convert each percent to a decimal.

9. 33%

10. 2.6%

11. 124%

12. 1240.5%

13. 42%

14. 4.5%

15. 0.003%

16. $\frac{1}{4}\%$

17. What is 12% of 72?

18. Ninety is what percentage of 200?

19. Twenty is 15% of what?

20. Six is what percentage of 40?

21. Forty is 91% of what?

22. What is 65% of 65?

23. In the fall of 2009 FCC enrolled 6,233 students. Of those enrolled, 2,810 are in the 18-21 age group. What percent of FCC students does this represent?

24. Patrick left an $8 tip on a $50 restaurant bill. What percent tip is that?

25. Ireland has a 23% VAT (value-added tax, similar to a sales tax). How much will the VAT be on a purchase of a €250 item?

26. Employees in 2012 paid 4.2% of their gross wages towards social security (FICA tax). How much would someone earning $45,000 a year pay toward social security?

27. A project on Kickstarter was aiming to raise $15,000 for a precision coffee press. They ended up with 714 supporters, raising 557% of their goal. How much did they raise?

28. Another Kickstarter project for an iPad stylus raised 1,253% of their goal, finishing with a total of $313,250 from 7,511 supporters. What was their original goal?

29. One year ago the median price for a home was $275,000. Now the current median price for a home is $235,000. What was the percent decrease in the median price of a home over the last year?

30. There were 943 tornadoes reported in the U.S. in 2013, and 897 tornadoes were reported in 2014. What percent decrease was there from 2013 to 2014?

31. The population of a town increased from 3,250 in 2008 to 4,300 in 2010. Find the absolute and percent increase.

32. The number of CDs sold in 2010 was 114 million, down from 147 million the previous year. Find the absolute and percent decrease.

33. A company wants to decrease their energy use by 15%.

(a) If their electric bill is currently $2,200 a month, what will their bill be if they're successful?

(b) If their next bill is $1,700 a month, were they successful? What percent decrease was there from the current bill?

34. A store is hoping an advertising campaign will increase their number of customers by 30%. They currently have about 80 customers per day.

(a) How many customers will they have if their campaign is successful?

(b) If they increase to 120 customers a day, were they successful? What percent increase is this from the current level?

35. An article reports that "attendance dropped 6% this year, to 300." What was the attendance before the drop?

36. An article reports that "sales have grown by 30% this year, to $200 million." What were sales before the growth?

37. The U.S. federal debt at the end of 2001 was \$5.77 trillion, and grew to \$6.20 trillion by the end of 2002. At the end of 2005 it was \$7.91 trillion, and grew to \$8.45 trillion by the end of 2006. Calculate the absolute and relative increase for 2001-2002 and 2005-2006. Which year saw a larger relative increase in federal debt?

38. A TV originally priced at \$799 is on sale for 30% off. There is then a 9.2% sales tax. Find the price after including the discount and sales tax.

39. The Walden University had 47,456 students in 2010, while Kaplan University had 77,966 students. Complete the following statements.

(a) Kaplan's enrollment was ____% larger than Walden's.

(b) Walden's enrollment was ____% smaller than Kaplan's.

(c) Walden's enrollment was ____% of Kaplan's.

40. In the 2012 Olympics, Usain Bolt ran the 100 m dash in 9.63 seconds. Jim Hines won the 1968 gold with a time of 9.95 seconds.

(a) Bolt's time was ____% faster than Hines'.

(b) Hines' time was ____% slower than Bolt's.

(c) Hines' time was ____% of Bolt's.

41. A store has clearance items that have been marked down by 60%. They are having a sale, advertising an additional 30% off clearance items. What percent of the original price do you end up paying?

42. A publisher marks up a textbook by 65%, and a bookstore further marks up the textbook by 35%. What percentage of the original cost do you pay?

SECTION 1.3 Simple and Compound Interest

Warren Buffett, in his regular letters to his partnerships, occasionally included a section titled "The Joys of Compounding," which he called "the pulse-quickening portion of our essay." In one such section in a letter from 1964, he wrote about the *Mona Lisa*:

> Since the whole subject of compounding has such a crass ring to it, I will attempt to introduce a little class into the discussion by turning to the art world. Francis I of France paid 4,000 ecus in 1540 for Leonardo da Vinci's *Mona Lisa*. On the off chance that a few of you have not kept track of the fluctuations of the ecu, 4,000 converted out to about $20,000.
>
> If Francis had kept his feet on the ground and he (and his trustees) had been able to find a 6% after-tax investment, the estate now would be worth something over $1,000,000,000,000,000.00. That's $1 quadrillion or over 3,000 times the present national debt, all from 6%. I trust this will end all discussion in our household about any purchase of paintings qualifying as an investment.

In a letter from the previous year, he concluded a similar example by saying

> Such fanciful geometric progressions illustrate the value of either living a long time, or compounding your money at a decent rate. I have nothing particularly helpful to say on the former point.

Don't worry, though; he gives a less fanciful example, as well, showing a table like the one below, which shows the returns from a $100,000 investment over 10, 20, and 30 years using different interest rates:

	4%	**8%**	**12%**
10 Years	$48,024	$115,892	$210,584
20 Years	$119,111	$366,094	$864,627
30 Years	$224,337	$906,260	$2,895,970

A quick glance through this table should make the point clear: compound interest is a powerful force (there's an apocryphal quote often attributed to Albert Einstein in which he calls it the most powerful force in the universe). Specifically, the power of compound interest comes from the length of time an investment is given to grow, and even small changes in the interest rate can have dramatic results given enough time.

By the way, although you won't find a bank account that offers interest rates anything close to the ones listed above, the long-term rate of return on the stock market is around 10%, so these values are by no means fictional.

These examples show what happens with *compound interest*, but we haven't yet discussed what that means.

Simple and Compound Interest

Simple Interest Interest is calculated based on the principal alone (the interest rate describes what percentage of the principal is returned).

Compound Interest The interest from one year (or month) is added to the principal, and the next year interest is calculated based on this combination of the principal and the past interest.

Simple Interest

Suppose you take out a loan for $500 at 10% annual interest rate for 4 years. Each year, $(\$500)(0.1) = \$50$ in interest accrues, so the total interest is 4 times this:

$$(\$500)(0.1)(4) = \$200$$

At the end of the 4 years, you'll have to pay back the principal, $500, plus the interest, $200, for a total of $700, so a present value of $500 grew to a future value of $700. Clearly, this growth depends on the interest rate and the amount of time involved.

Simple Interest

The interest, I, earned on a loan with principal P at annual interest rate r (expressed as a decimal) over a period of t years is

$$I = Prt$$

This formula works with other time periods (months, for instance) as long as the interest rate is given in the same terms (so a monthly interest rate, for instance).

Future Value The future value (F) of this principal (or present value) P is the sum of the principal and the interest:

$$F = P + Prt$$

$$F = P(1 + rt)$$

Future value for simple interest

Other kinds of loans (like compound interest) will have different formulas for future value, but the principal is the same: this formula tells at what rate this pile of money will grow.

APR: Annual Percentage Rate Note carefully that t is measured in *years*; this is consistent for almost all the financial formulas in this chapter. This means that interest rates are given as *annual* interest rates. It's also possible to express loans in monthly terms. To do so, the APR is divided into a monthly interest rate; for example, a 12% APR would be 1% monthly, a 6% APR would be 0.5% monthly, etc.

EXAMPLE 1 — SIMPLE INTEREST

Treasury notes and savings bonds are issued by the federal government to cover its expenses and debt. Suppose you obtain a $1,000 Series EE savings bond with a 4% annual rate and sell it 8 years later. How much interest will you earn?

Solution

Use the simple interest formula above:

$$\begin{aligned} I &= Prt \\ &= (\$1000)(0.04)(8) \\ &= \boxed{\$320} \end{aligned}$$

You'll earn $320 in interest, so at the end you'll have a total of $1320.

TRY IT

You deposit $3000 in a savings account at BB&T Bank, earning 5% interest. Find the amount of interest earned and the total amount in the account after three years.

When you're solving one of these problems, note carefully how the question is phrased. Some may ask for the interest earned by an investment, while others may ask for the total value of the investment at the end, which is simply the principal plus the interest.

EXAMPLE 2 FUTURE VALUE WITH SIMPLE INTEREST

If you deposit \$6200 at 6%, what is the future value of the deposit at the end of 2.5 years?

Solution Rather than calculating the interest first and adding that onto the principal, we can use the future value formula to do both steps at once:

$$\begin{aligned} F &= P(1+rt) \\ &= \$6200(1+(0.06)(2.5)) \\ &= \boxed{\$7130} \end{aligned}$$

TRY IT

What is the future value of a \$2400 investment at 7% simple interest at the end of three years?

We can also turn the problem around: if we know how much we want an investment to be worth in the future, we can use a little algebra to solve for the present value, or the principal.

EXAMPLE 3 PRESENT VALUE WITH SIMPLE INTEREST

You'd like to buy a \$12,000 car in 18 months, and your bank is offering 6% simple interest. How much should you deposit now in order to have a final balance of \$12,000?

We can use the same future value formula as in the previous example, but now the future value is given, and the present value is the unknown part:

$$\begin{aligned} F &= P(1+rt) \\ \$12,000 &= P(1+(0.06)(1.5)) \end{aligned}$$

Note that $t = 1.5$, since t is measured in years, and 18 months is one-and-a-half years.

Now solve this for P to find the amount that you need to deposit today:

$$\begin{aligned} \$12,000 &= P(1.09) \\ \frac{\$12,000}{1.09} &= P \\ \boxed{\$11,009.17} &= P \end{aligned}$$

You'll need to deposit \$11,009.17 today in order to have \$12,000 in the account in 18 months.

TRY IT

How much do you need to deposit today in an account earning 3% simple interest to have \$800 in 36 months?

Compound Interest

With simple interest, we assumed that we pocketed the interest when we received it. If, on the other hand, we added that interest to the account, we could earn interest on that interest in the future, making the balance grow a little bit faster. This reinvestment of interest is called **compounding**.

Suppose we deposit \$5000 for 5 years in an account offering an 8% APR, with interest compounded yearly. How much will be in the account at the end?

At the end of each year, 8% of the balance at that point will be added to the account, and the balance will grow. The following table shows on a year-to-year basis the total dollar amount in the account at the end of each year and the interest that accrues that year.

Year	Starting Balance	Interest Earned	Final Balance
1	\$5000	$\$5000 \times 0.08 = \$400$	$\$5000 + \$400 = \$5400$
2	\$5400	$\$5400 \times 0.08 = \$432$	$\$5400 + \$432 = \$5832$
3	\$5832	$\$5832 \times 0.08 = \$467$	$\$5832 + \$467 = \$6299$
4	\$6299	$\$6299 \times 0.08 = \$504$	$\$6299 + \$504 = \$6803$
5	\$6803	$\$6803 \times 0.08 = \$544$	$\$6803 + \$544 = \$7347$

The total amount in the account at the end of the fifth year is \$7347, which is \$347 more than we would have earned using simple interest.

Notice that each year, the amount of interest that we earned grew, making the account grow faster and faster. This is the advantage of compounding, and over long periods of time it can lead to very dramatic results.

Following this example, we can derive a formula to take care of the calculation for us so that we don't have to build a table like this every time. At the end of the first year, the balance had grown to

$$\$5000 + (\$5000)(0.08) = \$5000(1 + 0.08).$$

Following the pattern, each year the balance is multiplied by $(1 + 0.08)$, so at the end of the second year, the balance grew to

$$\$5000(1 + 0.08)(1 + 0.08) = \$5000(1 + 0.08)^2.$$

At the end of the third year, then, the account would hold

$$\$5000(1 + 0.08)^3$$

and so on.

After t years, the amount in an account with an interest rate of r, compounded once per year, will be

$$F = P(1 + r)^t$$

Future value for interest compounded yearly

What if interest isn't compounded yearly? The formula we just derived assumes that interest is compounded—or added to the account—at the end of each year. However, this doesn't have to be the case; interest could be compounded twice a year (semiannually), four times a year (quarterly), monthly, weekly, or even daily. keep track of how often interest is compounded, we define n as the **number of times per year that interest is compounded**, regardless of how many years the account grows.

Compounded	n
Annually	1
Semiannually	2
Quarterly	4
Monthly	12
Weekly	52
Daily	365[1]

Now if we split the year into n segments, the interest rate will be divided up as well, so each segment will earn an interest rate of $\frac{r}{n}$, so that will replace r in the compound interest formula. Also, rather than having the interest accrue t times, it will accrue n times each year for t years, or a total of nt times, so that will replace t in the compound interest formula.

All of this brings us to the complete formula for the future value of an investment with compound interest.

Compound Interest

The future value F of a principal amount P with an annual interest rate r (expressed as a decimal) compounded n times per year for t years is

$$F = P\left(1 + \frac{r}{n}\right)^{nt}$$

Notice that if interest is compounded yearly, $n = 1$, and the formula becomes the one we derived after the example above.

[1] Before calculators were commonplace, some calculations used $n = 360$ for daily compounding to make the arithmetic simpler

EXAMPLE 4 CERTIFICATE OF DEPOSIT

A certificate of deposit (CD) is an account that many banks offer that often comes with a higher interest rate, but the investment cannot be touched for a specified length of time. Suppose you deposit \$3000 in a CD earning 6% interest compounded monthly. How much will you be able to withdraw at the end of 20 years?

Solution List the pieces of the formula that are given:

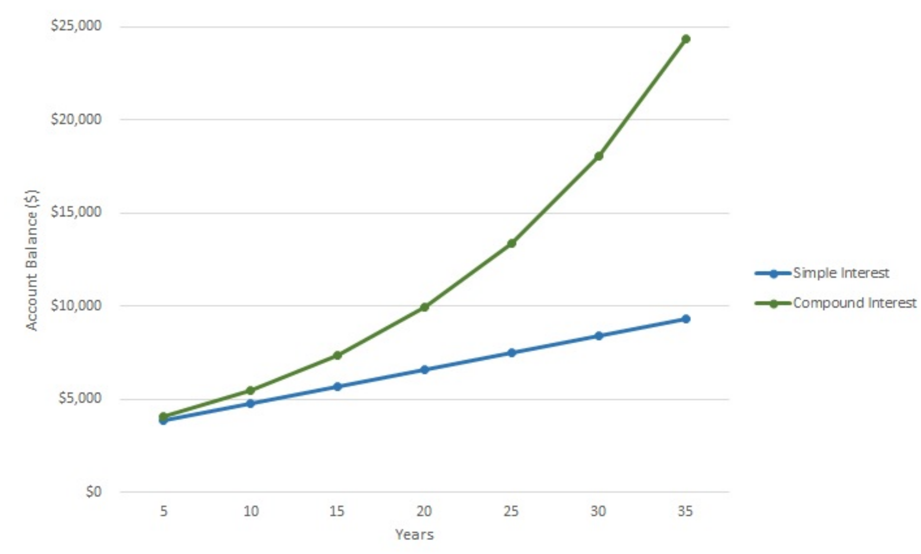
P	\$3000
r	0.06
n	12
t	20

So at the end of the 20 years, the account will hold

$$F = 3000\left(1 + \frac{0.06}{12}\right)^{(12)(20)} = \boxed{\$9930.61}$$

Now compare this to the amount you would earn from simple interest.

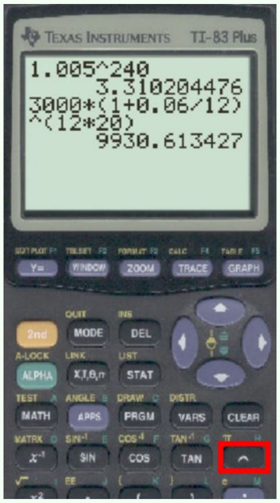
Years	Simple Interest	Compound Interest
5	\$3900	\$4046.55
10	\$4800	\$5458.19
15	\$5700	\$7362.28
20	\$6600	\$9930.61
25	\$7500	\$13,394.91
30	\$8400	\$18,067.73
35	\$9300	\$24,370.65



This illustrates the difference between the linear growth offered by simple interest and the exponential growth offered by compound interest. Over a long period of time, compounding makes a huge difference.

TRY IT If you deposit \$700 at 5% interest compounded monthly, how much will the account hold in 13 years?

Using Your Calculator: Exponents



To evaluate an exponent like 1.005^{240} we use the exponent key like the one shown, or possibly $\boxed{y^x}$ or $\boxed{x^y}$ on some calculators.

Be careful when evaluating these often-complicated financial formulas; it's usually safer to evaluate them in pieces, like in the first line, where we began by calculating $1 + 0.06/12 = 1.005$ and $(12)(20) = 240$, and then using the exponent key. If you want to evaluate the entire formula in one step, be careful to use parentheses to do each operation in the proper order, as shown in the second line.

Also, be very careful with rounding; keep at least three significant digits (digits after leading zeros) from one calculation to the next, or use the calculator storage function.

Just as we did with simple interest, we can also solve for present value with compound interest if we know what we want the future value to be.

SAVING FOR COLLEGE — EXAMPLE 5

You know that you will need $40,000 for your child's education in 18 years. If your account earns 4% compounded quarterly, how much would you need to deposit now to reach your goal?

Solution

List the pieces of the formula that are given:

F	$40,000
r	0.04
n	4
t	18

In this example, F is given and P is unknown:

$$40,000 = P\left(1 + \frac{0.04}{4}\right)^{(4)(18)}$$

Solve for P:

$$\begin{aligned} P &= \frac{40,000}{\left(1 + \frac{0.04}{4}\right)^{(4)(18)}} \\ &= \boxed{\$19,539.84} \end{aligned}$$

TRY IT

If you want to have $26,000 in a college fund in 12 years, and you find an account earning 5% compounded daily, how much should you deposit now?

Using Your Calculator: Avoid Rounding If You Can

In many cases, you can avoid rounding to make your answers more precise based on how you use your calculator. For example, to calculate something like

$$F = 1000\left(1 + \frac{0.05}{12}\right)^{(12)(30)}$$

start from the inside and work outward. We can quickly calculate (maybe even mentally) that $(12)(30) = 360$, and now we can use the calculator:

Type This	Calculator Shows
0.05 $\boxed{\div}$ 12 $\boxed{=}$	0.00416666666667
$\boxed{+}$ 1 $\boxed{=}$	1.00416666666667
$\boxed{y^x}$ 360 $\boxed{=}$	4.46774431400613
$\boxed{\times}$ 1000 $\boxed{=}$	4467.74431400613

EXAMPLE 6 DON'T ROUND TOO MUCH

To see why not over-rounding is so important, suppose you were investing \$1000 at 5% compounded monthly for 30 years.

P	\$3000
r	0.06
n	12
t	20

To use the formula, we'll need $\frac{r}{n}$, which is 0.0041666666666. . .

Notice the effect of rounding this to different values:

r/n rounded:	Gives F to be:	Error
No rounding	\$4467.74	
0.0041667	\$4467.80	\$0.06
0.004167	\$4468.28	\$0.54
0.00417	\$4473.09	\$5.35
0.0042	\$4521.45	\$53.71
0.004	\$4208.59	\$259.15

Notice that the error grew by *about* a factor of 10 each time, which is not unusual, considering that we rounded off a digit each time.

For our purposes, the answer we got by rounding to 0.004167 (four significant digits) is good enough - as long as we're not working in a bank, a rounding error of \$0.54 is fine for us.

Comparing Interest Rates

COMPARING BANKS

EXAMPLE 7

You have just won $500 in the Daily Pick 3 lottery and you decide to deposit your winnings in the bank. You check with two different banks, which offer different options. M&T Bank offers a 4.25% interest rate compounded daily, while SunTrust offers 4.3% compounded annually. Which bank should you choose?

To compare the two banks, simply choose an arbitrary amount of time and calculate how much each account would hold at the end of that time; whichever is higher is the one you'll choose. Let's pick a year as our length of time, just for simplicity. The table below shows the results of calculating the future value of your $500 at the end of a year with each bank.

M&T	SunTrust
$521.71	$521.50

Even though the difference is relatively small, you'll choose the first account, since over time the difference may grow to something more significant.

That example illustrates an important point: you'll often find different loans or accounts expressed in different terms, perhaps with different interest rates and compounding periods. In that case, you'll want to find some way to put them all on equal footing to compare them; like we did in that example, you can often do a quick calculation to see which will earn more in some arbitrary amount of time.

Banks will often take advantage of the financial illiteracy of their clients to present a loan in terms that will subtly benefit them. The major goal of this chapter is to turn you into a financially literate, savvy consumer.

Sidenote: Different Interest Rates

There are different terms you may find if you go looking for interest rates, so we'll mention a few of them here:

Nominal Interest Rate The *nominal interest rate* is the interest rate that is stated, such as a 3% annual rate on a bond or a 1.7% monthly rate on a credit card.

Annual Percentage Rate (APR) This is the annual nominal rate. So for instance, the 1.7% nominal monthly rate would correspond to a $1.7\% \times 12 = 20.4\%$ APR.

Effective Interest Rate (APY) This is the real interest rate. The *effective rate*, also called the *annual percentage yield (APY)* or the *effective annual yield*, takes the compounding period into account. This is done by calculating the simple interest rate that would lead to the same growth as that of the compound interest that is offered.

APR vs APY Since APY takes the effect of compounding into account and APR does not, the APY will be slightly higher than the APR for a typical account. Because of this, banks typically report the APR for debt-related accounts like credit cards and mortgages, and they report the APY for interest-bearing accounts like CDs and money market accounts.

Continuously Compounded Interest

We've seen that compound interest makes money grow faster than simple interest does, but we can go even further: if someone offered you an investment compounded monthly and one compounded daily (with everything else equal), which would you choose? You would be wise to choose the one that is compounded daily, because the more frequently that interest is compounded, the longer that interest that is added has to grow. In other words, the interest that is added after one day has more time to grow than if it had to wait until the end of the month to be added.

The question is this: is there a limit to this growth? Could we compound more and more often and see our money grow infinitely? To answer this, suppose we deposit \$1 for one year into an account with a fixed interest rate—we'll use 100% to illustrate, even though you'll almost certainly never encounter such an interest rate in real life—and we'll see what happens as we increase n, the frequency with which the interest is compounded:

n	$1\left(1+\frac{1}{n}\right)^n$
1	2.0000000...
5	2.4883200...
10	2.5937424...
50	2.6915880...
100	2.7048138...
1000	2.7169239...
10,000	2.7181459...
100,000	2.7182682...
1,000,000	2.7182804...
10,000,000	2.7182816...

Notice that as we compound more and more frequently, we start to approach a limit. This limit happens to be a very important number, so important that the letter e is reserved for it. It forms the basis of exponential growth, which has applications in every field of applied mathematics.

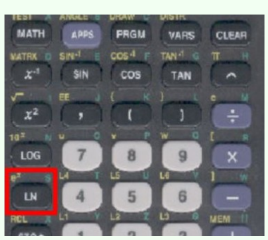
For a general interest rate r, $\left(1+\frac{r}{n}\right)^{nt}$ approaches e^{rt} as the compounding increases. This is what we call *continously compounded interest*.

Continuous Compound Interest

A present value P will grow to a future value of F under continuous compounding at an interest rate of r according to:

$$F = Pe^{rt}$$

Using Your Calculator: e



Your calculator will most likely have a button for e, but depending on what kind of calculator you have, it may look different.

Here, the model shown has e^x as the 2nd function of the button marked $\boxed{\text{LN}}$, so to calculate $e^{0.5}$, for instance, you would press $\boxed{\text{2nd}}$, then $\boxed{\text{LN}}$, then enter 0.5, and you should get 1.648721271.

On some scientific calculators, you may need to enter 0.5 and then press the $\boxed{e^x}$ key to get the same result.

CONTINUOUS COMPOUND INTEREST: FUTURE VALUE

EXAMPLE 8

If you deposit \$4500 in an account paying 3.2% interest compounded continuously, how much will the account hold after 36 months?

Solution

Use the continuous compound formula:

$$\begin{aligned} F &= Pe^{rt} \\ &= \$4500e^{(0.032)(3)} \\ &= \boxed{\$4953.42} \end{aligned}$$

TRY IT

If you deposit \$13,000 in an account paying 2.8% interest compounded continuously, how much will the account hold after 3 years?

Once again, we can also turn the problem around and solve for the present value.

CONTINUOUS COMPOUND INTEREST: PRESENT VALUE

EXAMPLE 9

How much will you need to deposit today at 5.3% compounded continuously in order to have \$6300 in 4 years?

Solution

Just like before, we'll use the same formula, but now F is known and P is the unknown part.

$$\begin{aligned} F &= Pe^{rt} \\ \$6300 &= Pe^{(0.053)(4)} \\ \$6300 &= P(1.2361) \\ \frac{\$6300}{1.2361} &= P \\ \boxed{\$5096.48} &= P \end{aligned}$$

TRY IT

How much will you need to deposit today at 3.6% compounded continuously in order to have \$1200 in 5 years?

EXAMPLE 10 COMPARING DIFFERENT COMPOUNDING PERIODS

You have \$7000 to invest for 5 years. Find how much you'll have at the end of the 5 years if you earn 4% compounded

(a) annually

(b) monthly

(c) daily

(d) continuously

Solution The first three all use the same formula, and all that changes is n:

(a) Compounded annually:

$$\begin{aligned} F &= P\left(1+\frac{r}{n}\right)^{nt} \\ &= 7000\left(1+\frac{0.04}{1}\right)^{(1)(5)} \\ &= \$8516.57 \end{aligned}$$

(b) Compounded monthly:

$$\begin{aligned} F &= P\left(1+\frac{r}{n}\right)^{nt} \\ &= 7000\left(1+\frac{0.04}{12}\right)^{(12)(5)} \\ &= \$8546.98 \end{aligned}$$

(c) Compounded daily:

$$\begin{aligned} F &= P\left(1+\frac{r}{n}\right)^{nt} \\ &= 7000\left(1+\frac{0.04}{365}\right)^{(365)(5)} \\ &= \$8549.73 \end{aligned}$$

(d) Compounded continuously: **note the different formula**

$$\begin{aligned} F &= Pe^{rt} \\ &= 7000e^{(0.04)(5)} \\ &= \$8549.82 \end{aligned}$$

TRY IT

If you deposit \$300, how much will you have in 7 years if you receive 3.5% interest compounded

(a) quarterly?

(b) monthly?

(c) weekly?

(d) continuously?

Notice how the future value increases as interest is compounded more and more frequently. The limit, of course, is the continuous case.

Doubling Time One common measure used to quickly compare investments is to determine how long it will take to double an investment. The shorter the doubling time, the better the investment.

Solving Exponential Equations

To solve an exponential equation, we use logarithms. For exponential equations, we'll use the *natural logarithm*, ln.

$$\text{If } e^x = y, \text{ then } x = \ln y$$

For more on solving equations involving exponents, see the algebra review chapter at the end of the book.

EXAMPLE 11 DOUBLING TIME

Find the time required to double an investment at 6% interest compounded continuously.

Solution

We could again pick an arbitrary amount for P, and let F be double that. Instead, though, we'll simply replace F with $2P$, and solve the formula for t:

$$\begin{aligned} F &= Pe^{rt} \\ 2P &= Pe^{0.06t} \\ 2 &= e^{0.06t} \\ \ln 2 &= 0.06t \\ \frac{\ln 2}{0.06} &= t \\ \boxed{11.55} &= t \end{aligned}$$

note the change from exponential form to logarithmic form here

Thus the investment will take approximately 11.5 years to double.

TRY IT

How long will it take an investment to double at 5% compounded continuously?

Doubling Time: the Rule of 72

A quick way to roughly approximate doubling time is to divide 72 by the percent interest rate (i.e. not the decimal form, but 100 times that)

$$\text{Doubling Time} \approx \frac{72}{R}$$

where $R = 100r$.

For continuous compounding, the *Rule of 69.3* is more precise, but 72 is an easier number to use for mental calculations, and other compounding terms will give results closer to the Rule of 72 than continuous compounding does.

So for example, with an interest rate of 6%, as in the last example, we could estimate the doubling time by dividing 72 by 6:

$$\frac{72}{6} = 12$$

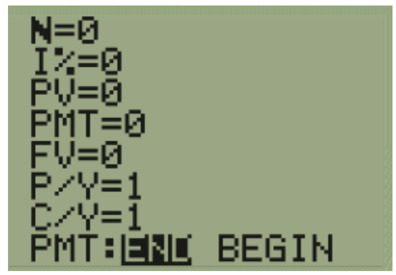
Even though this isn't as precise as the calculation in the example, notice how much simpler it was, and knowing that our investment will double in about 12 years is just as valuable as knowing it will be precisely 11.55 years.

Using The TVM Solver

If you have a TI-83 or TI-84 graphing calculator (or a similar model), you can use a built-in application that can solve questions involving compound interest without using the formulas in this section.

Although we'll show you how to use it here, and it is possible to answer the homework questions using it, you should make sure that you can solve the problems using the formulas, just to stretch your mathematical muscles.

To begin, find the button marked APPS . When you press that, you should have the option to select the Finance app, and when you press Enter twice, you'll enter the TVM (Time Value of Money) solver. You should then see a screen like the following one.



It's a good idea to zero out all the values at first (other than P/Y and C/Y), to make sure that you don't mistakenly use a value from another problem.

Let's go through the entries one at a time:

- `N`: the total number of compounding periods. This means, for instance, that if interest is compounded monthly for 8 years, N would be $(12)(8) = 96$. You could either calculate this separately, or simply type `12*8` in the space for N. Just remember that since there's no entry for t, this value N is equal to nt from our previous work.

- `I%`: the interest rate **as a percentage**, not a decimal. For instance, if the interest rate is 4.5%, simply enter 4.5 for I%.

- `PV`: the present value. Note that the TVM solver uses the sign to indicate which direction the money is moving; cash outflow is negative, and income is positive. If you deposit \$1000, for instance, you can enter -1000 for the present value, and the corresponding future value will be positive, since you'll recoup your investment.

- `PMT`: a recurring payment. This will come into play in the next section, but there's no need to use it for now; you can leave it as 0.

- `FV`: the future value. Again, the sign of this number will indicate which direction the money is moving.

- `P/Y`: payments per year. This too relates to the recurring payments that we'll encounter in the next section, but notice that it will always be the same as the next entry.

- `C/Y`: compounding periods per year. This is the familiar n we've been using in the compound interest formula.

You can ignore the option at the end; that toggles between payments made at the beginning and end of the payment period, but we're not getting that involved.

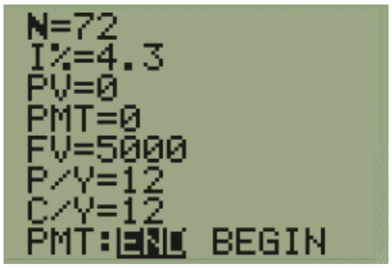
The way to use this application is to enter all the information that's given, then use the SOLVE option (one of the alternate functions of the ENTER key) to find the one that is unknown. We'll illustrate with an example.

USING THE TVM SOLVER — EXAMPLE 12

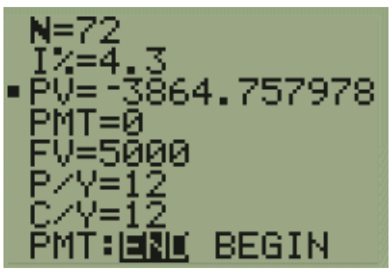
Use the TVM solver on your calculator to find the present value needed if we want a future value of $5000 in 6 years, if we can earn 4.3% interest compounded monthly.

Solution

Open the TVM solver and try entering all the information that is given. When you're done, you should see the following:



Now, move the cursor to hover over the 0 next to `PV=`. To access the SOLVE function on the ENTER key, first press the ALPHA key near the upper left corner of the keypad, then press the ENTER key.
When you do, you should see the following:



Notice that the present value is given as a negative number, which indicates that if you want to receive (positive) $5000 in the future, you need to deposit (negative) $3864.76 now.

You can also use the TVM solver, for instance, to calculate the necessary interest rate if you enter both a present value and a future value.

Of course, there's no magic to this application; it's simply using the formulas you have already been using, and you could solve any of these problems the long way. But it certainly is convenient to save yourself some typing.

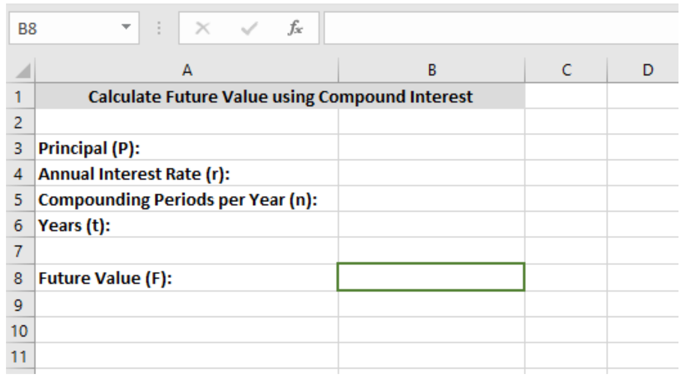
Sidenote: Continuous Compounding In case you're wondering how to handle continuous compounding, it turns out that it can be done, but it's a bit harder.

Since continuous compounding is essentially like interest compounding every instant, if we enter a huge number for C/Y, we can simulate this. In practice, something like 1 million works well enough. Then, let N be the number of years (so pretending that $n = 1$) and let P/Y be 1, and the result will be as close as we need it to be.

Using Excel

Although there are some built-in formulas in Excel to calculate things like future value, it turns out that it's just as easy to create the formulas yourself and build a simple compound interest calculator.

To begin, open Microsoft Excel and enter the following labels in the first column of a worksheet:



Notice that the highlighted cell (B8, since it is in column B and row 8) is where we want to find our answer. We'll place the inputs in cells B3-B6:

Value	Cell
P	B3
r	B4
n	B5
t	B6

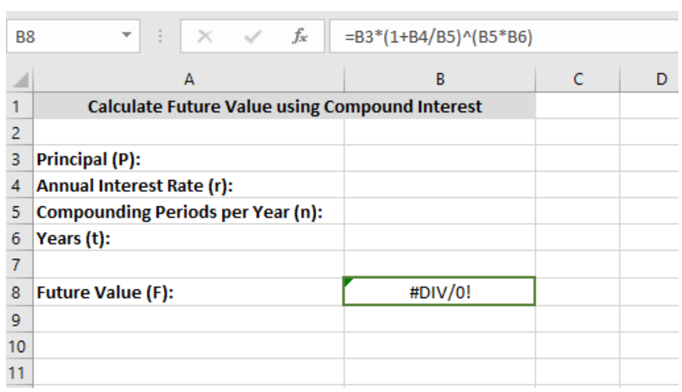
Thus, since the formula for F is

$$F = P\left(1 + \frac{r}{n}\right)^{nt}$$

we can simply replace each variable with the corresponding cell in the formula we type into cell B8:

Type "`=B3*(1+B4/B5)∧(B5*B6)`" into the formula box for cell B8.

Notice that we need to explicitly include a multiplication symbol when we want to multiply two values.



The result is an error at first, since we haven't filled in the other values, so the formula is forced to divide by 0, which of course isn't allowed.

Say, for instance, we want to find the future value of a \$2000 investment at 3% for 5 years, compounded monthly. Note that this time we'll use 0.03 for the interest rate, since the formula assumes r is in decimal form.

B8 =B3*(1+B4/B5)^(B5*B6)

	A	B	C	D
1	**Calculate Future Value using Compound Interest**			
2				
3	**Principal (P):**	2000		
4	**Annual Interest Rate (r):**	0.03		
5	**Compounding Periods per Year (n):**	12		
6	**Years (t):**	5		
7				
8	**Future Value (F):**	\$ 2,323.23		
9				
10				
11				

Notice that the output can be nicely formatted as a dollar amount by opening the Format Cells option after right-clicking on the cell.

There are many more ways to use Excel as a financial tool, but to calculate the result of compound interest is simple enough, using the familiar formula.

Exercises 1.3

In problems 1–3, a principal amount is borrowed at the given interest rate for the given period of time. Find the loan's future value F, or the amount due at the end of the time, if the loan uses simple interest.

1.

Principal:	\$3000
Interest rate:	7%
Time:	2 years

2.

Principal:	\$2700
Interest rate:	4%
Time:	3 years

3.

Principal:	\$7500
Interest rate:	3.5%
Time:	18 months

In problems 4–6, a principal amount is borrowed at the given interest rate for the given period of time. If the future value is given, find the principal (present value) if the loan uses simple interest.

4.

Future value:	\$9000
Interest rate:	5.5%
Time:	1 year

5.

Future value:	\$7700
Interest rate:	6%
Time:	4 years

6.

Future value:	\$800
Interest rate:	2.75%
Time:	9 months

In problems 7–9, a principal amount is borrowed at the given interest rate for the given period of time, and interest is compounded as stated. Find the loan's future value F, or the amount due at the end of the time.

7.

Principal:	\$1200
Interest rate:	5%
Compounding:	Annually
Time:	3 years

8.

Principal:	\$5700
Interest rate:	3.5%
Compounding:	Monthly
Time:	24 months

9.

Principal:	\$3000
Interest rate:	5.32%
Compounding:	Continuously
Time:	48 months

In problems 10–12, a principal amount is borrowed at the given interest rate for the given period of time, and interest is compounded as stated. If the future value is given, find the principal.

10.

Future value:	\$17,500
Interest rate:	3%
Compounding:	Annually
Time:	8 years

11.

Future value:	\$18,000
Interest rate:	5.6%
Compounding:	Daily
Time:	18 months

12.

Future value:	\$9000
Interest rate:	7.48%
Compounding:	Continuously
Time:	60 months

13. A friend lends you \$200 for a week, which you agree to repay with 5% one-time interest. How much will you have to repay?

14. Suppose you obtain a \$3,000 T-note with a 3% annual rate, paid quarterly, with maturity in 5 years. How much interest will you earn?

15. A student took out a simple interest loan for \$2,400 for two years at an annual rate of 7%.

(a) What is the interest on the loan?

(b) How much will the student have to repay at the end of two years?

16. A loan of \$2,040 has been made at a 5.7% annual simple interest rate for four months.

(a) What is the interest on the loan?

(b) Find the future value of the loan.

17. You deposit \$2000 in an account earning 3% interest compounded monthly.

(a) How much will you have in the account in 20 years?

(b) How much interest will you earn?

18. You deposit \$10,000 in an account earning 4% interest compounded weekly.

(a) How much will you have in the account in 25 years?

(b) How much interest will you earn?

19. How much would you need to deposit in an account earning 6% compounded monthly in order to have \$6,000 in the account in 8 years?

20. How much would you need to deposit in an account earning 5% compounded quarterly in order to have \$20,000 in the account in 4 years?

21. If you deposit \$5400 in an account earning 4.35% interest compounded continuously, how much will the account hold in 18 months?

22. If you take out a loan for \$7700 at 6.7% interest compounded continuously, how much will you have to pay back in 5 years?

23. How much do you need to deposit today at 4% interest compounded continuously in order to have \$4000 in 2 years?

24. If you find a CD offering 5.8% interest compounded continuously, how much should you deposit if you are saving up to refinish your kitchen in 3 years and you estimate that will take \$15,000?

25. You have \$12,000 to invest for 3 years. Find how much you'll have at the end of the 3 years if you earn 4% interest compounded

(a) annually

(b) monthly

(c) daily

(d) continuously

26. You would like to have \$8000 saved in 3 years. Find how much you'll have to invest now to reach that goal if you earn 6% interest compounded

(a) annually

(b) monthly

(c) daily

(d) continuously

27. How long will it take to double an investment at 7% compounded continuously?

28. How long will it take to double an investment at 4.6% compounded continuously?

SECTION 1.4 Saving for Retirement

Who wants to be a millionaire? More realistically, who *doesn't* want to be a millionaire? It may sound like an unattainable goal, but in this section, you'll learn how you can join the millionaire club by taking advantage of the power of compound interest.

If you forget everything else you learn here, don't forget this one lesson:

START SAVING EARLY

The results speak for themselves:

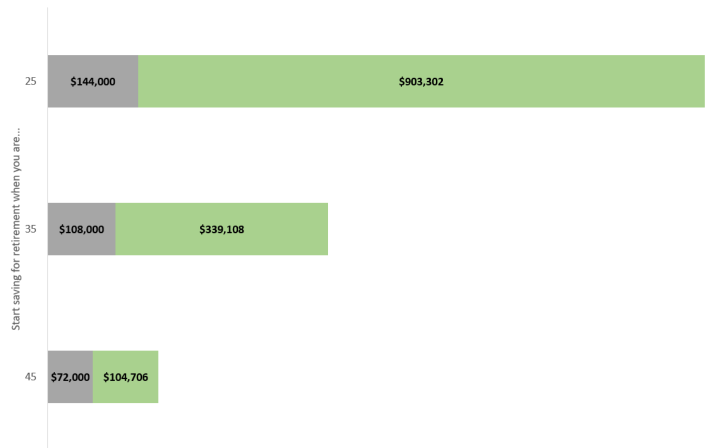
This graph shows how much you can accumulate if you can save $300 a month, assuming that you receive an 8% annual return on your investments, a fairly typical result if, for instance, you invest in index funds, which are mutual funds designed to track the market as a whole over the long term.

That may sound like a lot to save, but this is about the amount that many people spend on a car payment. What this means is that with relatively small sacrifices, such as driving used cars instead of new ones, you can someday see your retirement account cross the $1,000,000 threshold.

Let's dig into that chart a bit more. Notice that if you wait until age 35 to start saving, you won't just save a *bit* less than you would if you started saving earlier; you'll wind up with *less than half*, just by waiting 10 years. Maybe you can't afford to save $300 a month right now, but if you can afford $100 a month, start there!

Why is there such a dramatic difference? It's all due to compound interest, and we learned in the last section that compound interest really flexes its muscles when you give it plenty of time. Those 10 years between ages 25 and 35 are the most important years in the process, because they are the first years, and the money saved then has the most time to grow.

To show this, let's break down the amount that you will actually deposit in each case, versus the amount you'll earn in interest:



Notice that if you wait until age 45 to start saving, you'll end up depositing half of what you would if you started at 25, but your final balance will be about *one sixth* of what it could be. Another way to look at it is that starting at age 25 multiplies your money by over seven times (the ratio of what you actually save and what interest adds), but starting at age 45 only multiplies your money by about two and a half.

In this section, you'll learn how all of those values were calculated, but the conclusion should already be clear: don't wait any longer to start saving!

Annuities

The term *annuity* simply refers to a series of recurring payments. This is different from what we did in the last section, where we assumed that a single lump-sum deposit was made and a single withdrawal was made later. We haven't yet dealt with accounts that take regular deposits, but that's what you do for any kind of long-term savings plan: use small, regular payments to grow over time.

In this section, we'll discuss two types of annuities, which are similar, but the formulas are slightly different.

Types of Annuity

Savings Annuity This is the kind that you use to save for retirement; you make regular deposits over time, and the balance slowly grows because of two factors: your deposits and the accruing interest.

Payout Annuity When you retire, you stop making payments into your retirement plan, and you start receiving regular payouts instead, which replace your paycheck. In this case, we start the problem with a lump sum (what you managed to save) and compare that lump sum with the payment amount that you receive.

If you do more research on annuities, you'll find that there are other variations, such as lifetime annuities, or that annuities apply in other situations (usually some large payment such as life insurance or lottery winnings can be paid out using annuities), but our focus will be on retirement savings, since that's the most common scenario.

For our purposes, the basic process looks something like this: starting today, you can deposit money into a savings annuity, and we can calculate what your final balance will be when you retire. Then, we can fast-forward to the day of retirement, and projecting how much your account will hold, we can find out how much you can expect to receive as a monthly payment in retirement. We can also work backward from how much you'll need in retirement to calculate the balance you need when you retire, then work backward again to calculate how much you'll need to start saving.

Savings Annuities

Let's start with a simple example to illustrate how a savings annuity works.

EXAMPLE 1 SAVINGS ANNUITY

Suppose you deposit \$100 into a savings account at the end of each year. If you earn 5% interest compounded annually, how much will the account hold at the end of 3 years? How much interest did the account earn?

Solution At the end of the first year, the account holds the \$100 that you deposit then:

$$F_1 = \$100$$

The second year, this \$100 earns interest, plus you deposit another \$100 at the end of the year:

$$F_2 = \$100(1 + 0.05) + \$100 = \$205$$

The third year, this \$205 earns interest, plus you deposit another \$100 at the end of the year:

$$F_3 = \$205(1 + 0.05) + \$100 = \$315.25$$

We could continue this pattern indefinitely, but each year, we only need to use the simple interest formula to see how much the previous year's balance has grown, and then add in that year's deposit.

At the end of the three years, the account holds \$315.25, and since we deposited a total of \$300 (\$100 each year for 3 years), the account earned a total of \$15.25 in interest.

In theory, we could do that every time we need to calculate the value of an annuity, but that would get tedious very quickly. The following formula is derived based on that process (details are at the end of this section, in case you'd like to see the algebra).

The Future Value of a Savings Annuity

If regular deposits of PMT are made n times per year into an annuity paying an interest rate of r compounded n times per year, the future value of the annuity at the end of t years is given by

$$F = \frac{PMT\left[\left(1 + \frac{r}{n}\right)^{nt} - 1\right]}{\left(\frac{r}{n}\right)}$$

This formula looks complicated, but with a little practice, you'll be able to use it, for instance, to verify the numbers in the retirement discussion at the beginning of the section.

By the way, we may also start by determining how much we want the account to hold at the end, and use that to calculate PMT. To do that, we need to solve the formula above for PMT, which means multiplying both sides by r/n and dividing both sides by the expression in brackets:

$$PMT = \frac{F\left(\frac{r}{n}\right)}{\left(1 + \frac{r}{n}\right)^{nt} - 1}$$

You can choose to keep this formula handy as well, or you can simply do the algebra each time. It helps to enter all the values and simplify as much as possible on the right hand side, and then simply divide F by that simplified value.

EXAMPLE 2 TRADITIONAL IRA

A traditional individual retirement account (IRA) is a retirement account in which the money you invest is tax-exempt (you can deduct your contributions on your income tax return) until you withdraw it. Thus, taxes are deferred until you retire. If you deposit \$250 each month into an IRA earning 7% interest, how much will you have in the account after 35 years?

Solution

Organize the given information:

PMT	\$250	The regular deposit
r	0.07	7% annual rate
n	12	Deposits are made monthly
t	35	Deposits are made for 35 years

Putting it all together in the formula:

$$F = \frac{\$250\left[\left(1+\frac{0.07}{12}\right)^{(12)(35)} - 1\right]}{\left(\frac{0.07}{12}\right)} = \boxed{\$450,264}$$

Notice that you deposited \$250 every month, 12 months a year for 35 years, for a total of \$105,000. That means that the account earned approximately $\$450,264 - \$105,000 = \$345,264$ in interest, or to put it another way, the deposits more than quadrupled due to interest.

$(\$250)(12)(35) = \$105,000$

TRY IT

If you deposit \$800 every year into a traditional IRA earning 4% interest, how much will the account hold after 25 years?

There is another common type of retirement account: the Roth IRA. The idea behind a Roth IRA was originally proposed in 1989 by Senator William Roth of Delaware and established by the Taxpayer Relief Act of 1997.

The differences between traditional and Roth IRAs do not affect the calculations in the examples, but you may hear both terms discussed, so we'll include a short comparison here just for your interest (you can skip this if you prefer).

Traditional vs. Roth IRA

The main difference between the two types of IRA (individual retirement accounts) revolves around taxes. With a traditional IRA, contributions are tax-free (you can deduct these on your tax return) and you pay taxes when you withdraw the funds in retirement. Roth IRA contributions, on the other hand, are taxed when you make them, but when you make withdrawals after retiring, you won't pay taxes then.

Thus, when comparing the two, the question is: do you expect to pay higher taxes now, or when you are retired? It's hard to know for sure, because your income may be higher or lower than it is now, and many of the deductions you'll take advantage of in the near term (like mortgage insurance, dependent expenses, or education costs) will not apply when you are retired.

There are other qualitative differences, like the fact that a traditional IRA will require you to start withdrawing when you reach 70.5 years of age; Roth IRAs don't have that restriction, making them a popular vessel for transferring wealth to inheritors.

Contribution Limits (2020) The most you can contribute to either kind of IRA (or a combination of the two) in a single year is \$6,000 if you're younger than 50, or \$7,000 if you're 50 or older.

Income Limits (2020) Traditional IRAs have no income limits, but you can't contribute to Roth IRAs if you make too much money. For instance, the current limit for single individuals is \$124,000, and \$196,000 for married couples; if your adjusted income is higher, you can only contribute to a traditional IRA that year.

One final note: both kinds of IRA allow first-time homebuyers to withdraw up to \$10,000 to pay for qualified housing costs.

We've seen an example of calculating F given PMT; let's switch things around and look for the payment amount if we know how much we want the account to hold.

EXAMPLE 3 HOW MUCH SHOULD YOU SAVE?

You want to have $500,000 in your account when you retire in 35 years. If your retirement account earns 5% interest, how much should you deposit each month to reach your retirement goal?

Solution Now everything except for PMT is given, and that is what we are trying to determine.

F	$500,000	The future value
r	0.05	5% annual rate
n	12	Deposits are made monthly
t	35	Deposits are made for 35 years

Putting it all together in the formula:

$$\$500,000 = \frac{PMT\left[\left(1+\frac{0.05}{12}\right)^{(12)(35)}-1\right]}{\left(\frac{0.05}{12}\right)}$$

$$\$500,000 = PMT(1136.092)$$

$$\boxed{\$440.11} = PMT$$

$(\$440)(12)(35) = \$184,846.20$

$\$500,000 - \$184,846 = \$315,154$

Having half a million dollars may sound like an unattainable goal, but by making regular deposits, it becomes possible. Notice that in this case, you'll deposit a total of $184,846, which means that the account earns $315,154, or close to twice the amount that you deposit.

We could also have used the alternate form of the formula where PMT is isolated, but this example illustrates how to simplify and solve for PMT by using the same formula as before. It's up to you which process you prefer: a bit more algebra, or keeping track of another formula.

TRY IT

You want to have $800,000 in your account when you retire in 40 years. If your retirement account earns 6.7% interest, how much should you deposit each month to reach your retirement goal?

In case the discussion at the beginning of the section wasn't enough, let's look at another example emphasizing the same point with a slightly different scenario. We'll consider two recent college graduates. Emma learns her lesson and begins saving immediately, while Jason is overwhelmed by his expenses immediately as he begins to work and he neglects to save. After 20 years, Jason decides to try to catch up. Let's see how that works out (spoiler alert: Emma winds up better off).

Suppose Jason and Emma graduate the same year and begin working in adjacent cubicles; they're each 23 years old, and they'll both work for 45 years. Let's assume that both get a 7% interest rate in their retirement accounts.

EXAMPLE 4 START SAVING EARLY

(1) If Emma begins saving $400 every month right away and does so for 45 years, how much will her account hold when she retires?

Use the savings annuity formula:

$$F = \frac{\$400\left[\left(1+\frac{0.07}{12}\right)^{(12)(45)}-1\right]}{\left(\frac{0.07}{12}\right)}$$

$$\approx \boxed{\$1,517,038}$$

(2) If Jason begins saving 20 years later, and he saves \$1000 every month for 25 years, how much will his account hold when he retires?

Use the savings annuity formula:

$$F = \frac{\$1000\left[\left(1+\frac{0.07}{12}\right)^{(12)(25)}-1\right]}{\left(\frac{0.07}{12}\right)}$$
$$\approx \boxed{\$810,072}$$

Notice that he has to save more than her (\$300,000 versus \$216,000), but he winds up way behind, because his savings didn't have as much time to grow.

(3) How much would Jason have to save each month for 25 years to match Emma's final total?

To find this out, set F equal to Emma's final total, and solve for PMT using $t = 25$:

$$\$1,517,038 = \frac{PMT\left[\left(1+\frac{0.07}{12}\right)^{(12)(25)}-1\right]}{\left(\frac{0.07}{12}\right)}$$
$$\boxed{\$1873} \approx P$$

He'd have to save much, much more each month to catch up to Emma, and in total this would mean saving \$561,816 (again, compared to the \$216,000 that Emma saves in total).

(4) Compare their contributions to their final balances.

(a) Emma contributes a total of $\$400 \times 12 \times 45 = \$216,000$, which means that her account earned \$1,301,038 in interest.

(b) Under Jason's first plan, he contributes \$300,000 (more than Emma, even though his final balance is much smaller), so he only earns \$510,072 in interest.

(c) With Jason's modified plan where he contributes \$1873 each month, he pays in a total of \$561,816, earning \$955,222 in interest. This interest is so large, though, because of the large payments he makes, which grow the balance quickly, and not due to the length of time given for growth, which is where compound interest really shines.

This lesson bears repeating: start saving early!

Payout Annuities

We're ready now to shift from savings annuities, where you make regular deposits to save toward a final lump sum, to payout annuities, in which that lump sum is repaid to you in regular payments and the remaining balance continues to earn interest in the meantime. Once again, this is typically the shift that occurs upon retirement, and it's important to understand in order to intelligently plan for retirement.

As with savings annuities, we'll start with the formula (the derivation for this one can also be found at the end of the section).

Payout Annuities

If a starting balance of P is paid out in regular payments of PMT from an annuity earning r interest compounded n times per year, and the payments are made n times per year, the following relationship holds:

$$P = \frac{PMT\left[1-\left(1+\frac{r}{n}\right)^{-nt}\right]}{\left(\frac{r}{n}\right)}$$

Notice the negative exponent; be careful when entering that into your calculator.

Notice that the formula is set up to solve for P rather than PMT; this is mostly for convenience, because when planning for retirement, you usually start by deciding how much you need to receive every month to cover your needs. We can, however, turn the problem around: if we start knowing (or assuming) how much the account will hold at the beginning, we can solve for PMT just as we did with savings annuities.

$$PMT = \frac{P\left(\frac{r}{n}\right)}{1-\left(1+\frac{r}{n}\right)^{-nt}}$$

Again, you could keep track of this formula as well, or simply solve for PMT whenever needed.

EXAMPLE 5 PAYOUT ANNUITY

After retiring, you want to be able to take \$1000 every month from your retirement account for 20 years. If the account earns 6% interest, how much will you need in your account when you retire?

Solution Organize the given information:

PMT	\$1000	The regular withdrawal
r	0.06	6% annual rate
n	12	Withdrawals are made monthly
t	20	Withdrawals are made for 20 years

Putting it all together in the formula:

$$\begin{aligned} P &= \frac{\$1000\left[1-\left(1+\frac{0.06}{12}\right)^{-(12)(20)}\right]}{\left(\frac{0.06}{12}\right)} \\ &\approx \boxed{\$139,581} \end{aligned}$$

You'll need to have approximately \$139,600 in your account when you retire. Notice that you'll withdraw \$240,000 (\$1000 for 240 months). You're able to pull out more than you have at retirement because you don't withdraw it all at once, but take it out little by little as you need it, allowing the remainder to earn interest before you take it out. This difference represents \$100,400 in interest earned during those 20 years of retirement.

TRY IT

After retiring, you want to be able to take \$1500 every month from your retirement account for 15 years. If the account earns 4.5% interest, how much will you need in your account when you retire?

Calculator Note: Evaluating Negative Exponents

With these problems, you need to raise numbers to negative powers. Most calculators have a separate button for negating a number that is different than the subtraction button. Some calculators label this $\boxed{(-)}$, and some label it $\boxed{+/-}$.

If your calculator has a multiline display, to calculate 1.005^{-240}, you'd type something like 1.005 $\boxed{\wedge}$ $\boxed{(-)}$ 240.

If you have a scientific calculator that only displays a single number at a time, you will most likely need to hit the $\boxed{(-)}$ key after a number to negate it. Thus, you'd type 1.005 $\boxed{y^x}$ 240 $\boxed{(-)}$ $\boxed{=}$.

Try it on your calculator and make sure that you get 0.302096 as your answer.

Finally, let's turn this around and ask the other question: given a fixed amount in our account, how much can we withdraw in regular payments?

WITHDRAWING FROM A PAYOUT ANNUITY

EXAMPLE 6

You expect to have \$500,000 in your IRA when you retire, and you want to be able to take monthly withdrawals for a total of 30 years. If your account earns 8% interest, how much will you be able to withdraw each month?

Solution

Organize the given information:

P	\$500,000	The starting balance
r	0.08	8% annual rate
n	12	Withdrawals are made monthly
t	30	Withdrawals are made for 30 years

This time we want to find PMT:

$$\$500,000 = \frac{PMT\left[1-\left(1+\frac{0.08}{12}\right)^{-(12)(30)}\right]}{\left(\frac{0.08}{12}\right)}$$

$$\$500,000 = PMT(136.232)$$

$$\boxed{\$3670} \approx PMT$$

Note: if you don't round at this step, your answer should be \$3668.82

You can plan to withdraw about \$3670 each month for 30 years.

TRY IT

A donor gives \$100,000 to a university, and specifies that it is to be used to give annual scholarships for the next 20 years. If the university can earn 4% interest, how much can they give in scholarships each year?

Let's put it all together, and try a full example in which we start with the desired monthly payout during retirement, and determine how to start saving now. This is the kind of planning that you can do right now for yourself, using the tools we've discussed in this section.

EXAMPLE 7 PLANNING FOR RETIREMENT

Kevin is 30 years old, and he is preparing to begin saving for retirement. He expects to retire at age 67, and for planning purposes, he assumes he'll live to age 95. Based on cursory research, he expects that his investments can average a return of 7% annually, and after retirement, he will move his money into more conservative investments returning 5% annually. In order to be able to withdraw \$3000 per month after retirement, how much should he plan to save each month?

Solution

There are two stages to this problem; it's essentially like doing two of the previous examples back-to-back. First, we need to use the payout annuity formula to find how much Kevin's retirement account must hold at age 67 to meet his goals. Once we know that, we can work backward using the savings annuity formula to find the payment amount that will lead to that future value.

1. Find balance at retirement

For the first stage, using the payout annuity formula, the following summarizes what we know:

PMT	\$3000	The regular withdrawal
r	0.05	5% annual rate
n	12	Withdrawals are made monthly
t	28	Withdrawals are made for 28 years

Putting it all together in the formula:

$$\begin{aligned} P &= \frac{\$3000\left[1-\left(1+\frac{0.05}{12}\right)^{-(12)(28)}\right]}{\left(\frac{0.05}{12}\right)} \\ &\approx \$541,933 \end{aligned}$$

At age 67, then, Kevin's retirement account needs to hold \$541,933. Knowing that, we can now shift focus to the savings annuity that he will use between now and age 67 to accumulate that total.

2. Find savings payment

Notice that now the interest rate will change to 7%, and the amount of time to save will be 37 years (from age 30 to age 67):

F	\$541,933	The future value
r	0.07	7% annual rate
n	12	Deposits are made monthly
t	37	Deposits are made for 37 years

Using the savings annuity formula:

$$\begin{aligned} \$541,933 &= \frac{PMT\left[\left(1+\frac{0.07}{12}\right)^{(12)(37)}-1\right]}{\left(\frac{0.07}{12}\right)} \\ \boxed{\$258.49} &= PMT \end{aligned}$$

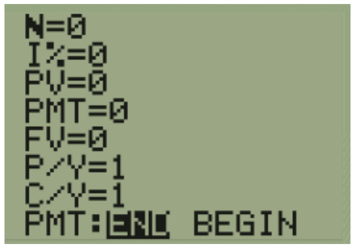
By setting aside a little over \$250 each month into a retirement account, Kevin can build quite a decent nest egg, and ensure that he'll be able to withdraw \$3000 a month after he retires.

TRY IT

Erika is 27 years old, and plans to retire at age 64. If she can expect an 8% return while she's saving, and a 6% return while she withdraws, how much should she begin saving each month if she expects to live to age 92 and would like to receive \$3500 a month in retirement?

Using the TVM Solver

We can use the TVM solver that was introduced for compound interest in the previous section to solve problems with annuities as well. Recall the setup of the solver (found under the APPS menu):



Notice the entry for PMT (on the fourth line), which we did not use last time. This is the same as the payment amount PMT in our formulas in this section.

Savings vs. Payout Annuities The main difference to keep in mind when using the TVM solver is that for a savings annuity, we're using payments to build to a *future value* FV, and with a payout annuity, we have a *present value* PV that gets paid out in regular payments.

So for a problem involving a savings annuity, set PV to 0 and use FV and PMT (find whichever we need to based on knowing the other), and for a payout annuity, set FV to 0 and use PV and PMT.

The rest of the information in the menu is the same as before. In almost every problem, if not every single one, payments occur monthly, so P/Y, and C/Y will both be 12. Recall that I% should be entered in percentage form, not decimal form, and you should be all set!

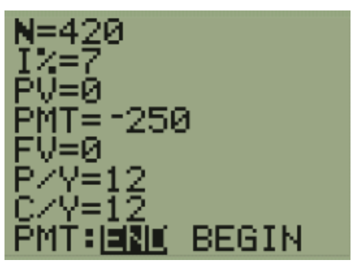
We'll show one example for each kind of annuity; both will be repeated from earlier examples for comparison.

TVM SOLVER: FUTURE VALUE FOR SAVINGS ANNUITY — EXAMPLE 8

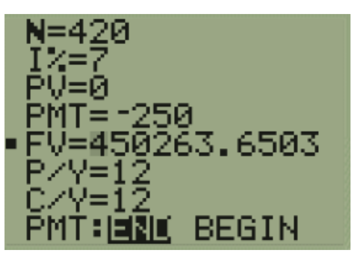
If you deposit $250 each month into an IRA earning 7% interest, how much will you have in the account after 35 years?

Solution

Open the TVM solver and enter the information given (remember that N is the total number of payments, so enter `12*35`. When you're done, you should have the following:



To solve for the future value, move the cursor over the value for FV, and press ALPHA then ENTER to solve. You should see the answer entered there:

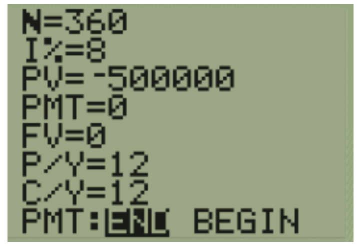


The final balance will be $450,264 (verify that this matches what we found using the formula earlier).

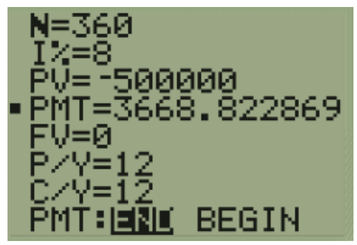
EXAMPLE 9 TVM SOLVER: PAYMENT FOR PAYOUT ANNUITY

You expect to have $500,000 in your IRA when you retire, and you want to be able to take monthly withdrawals for a total of 30 years. If your account earns 8% interest, how much will you be able to withdraw each month?

Solution Here's the setup (remember, set FV to 0 and work with PV and PMT):



Remember that we use negative numbers to indicate money we spend or save, and positive for money we receive (if you forget to do this during setup, it doesn't cause any problems; just mentally reverse the signs). After solving, you should see this:



Notice that the answer is slightly different from the one we got using the formula (within a dollar or two), because we rounded partway through that problem.

TRY IT Try using the TVM solver to work out the other examples in this section.

Using Excel

Excel has built-in formulas to solve the same problems:

Finding the payment `PMT(rate,nper,pv,[fv],[type])`

Finding the future value for a savings annuity `FV(rate,nper,pmt,[pv],[type])`

Finding the present value for a payout annuity `PV(rate,nper,pmt,[fv],[type])`
In each formula, the items in brackets are optional, and we won't use them here (in case you're curious, the "type" is the same as the option at the bottom of the TVM solver to switch between beginning-of-month and end-of-month payments).

Important note: the rate used in these formulas is the **monthly** rate, or the given annual interest rate divide by 12. Also, nper refers to the number of periods, which is nt in our formulas.

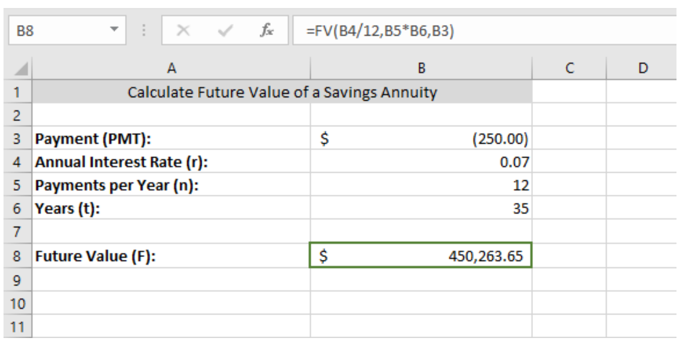
We'll show one example here: calculating the future value of a savings annuity.

EXCEL: FUTURE VALUE FOR SAVINGS ANNUITY — EXAMPLE 10

If you deposit $250 each month into an IRA earning 7% interest, how much will you have in the account after 35 years?

Solution

Here's the result in Excel:



B8 =FV(B4/12,B5*B6,B3)

	A	B	C	D
1	Calculate Future Value of a Savings Annuity			
2				
3	Payment (PMT):	$ (250.00)		
4	Annual Interest Rate (r):	0.07		
5	Payments per Year (n):	12		
6	Years (t):	35		
7				
8	Future Value (F):	$ 450,263.65		
9				
10				
11				

Note that the payment amount has parentheses around it; in accounting worksheets, this is how negative values are often indicated. The convention here is the same as with the TVM solver; since these are payments going out, we count them as negative.

Formula: in cell B8, we entered `=FV(B4/12,B5*B6,B3)`. Remember that the rate should be the monthly interest rate, which is why we divided the given annual rate by 12, and the value of nper is equal to the number of payments per year times the number of years.

TRY IT

Try using Excel to work out the other examples in this section.

Deriving the Formulas (optional)

In case you're *very* curious, and would like to see how the formulas for savings and payout annuities are derived, you can read this. You can freely skip this part, though, without missing anything major.

Savings Annuity Formula Suppose you deposit P dollars (we'll rename this PMT at the end) into a savings annuity each year, and this account earns an interest rate of r compounded annually (we'll handle the case of other compounding periods after we get to the formula). At the end of the first year, the account contains P dollars:

$$F_1 = P$$

This principal earns interest the second year [growing to $P(1+r)$] so at the end of the second year, the account holds that plus the newly deposited P:

$$F_2 = P + P(1+r)$$

Now, in the third year, this balance earns interest again: $(P + P(1+r))(1+r) = P(1+r) + P(1+r)(1+r)$, so the balance at the end of the third year is this plus another P:

$$F_3 = P + P(1+r) + P(1+r)^2$$

We can now see the pattern, so we can jump to the arbitrary case; at the end of t years, the account will hold

$$F_t = P + P(1+r) + P(1+r)^2 + P(1+r)^3 + \ldots + P(1+r)^{t-1} \tag{1.1}$$

Now comes the tricky part: we want a simpler formula for F_t, so we solve for it in an unexpected way. First, multiply both sides of the last line by $(1+r)$:

$$F_t + F_t r = P(1+r) + P(1+r)^2 + P(1+r)^3 + \ldots + P(1+r)^{t-1} + P(1+r)^t \tag{1.2}$$

Next, subtract equation (1.1) from equation (1.2), subtracting on both sides of the equation. Notice that as we do so, almost all of the terms cancel:

$$F_t r = P(1+r)^t - P = P\left[(1+r)^t - 1\right]$$

Finally, divide both sides of the equation by r to isolate F_t:

$$F_t = \frac{P\left[(1+r)^t - 1\right]}{r}$$

What if we make deposits monthly rather than yearly? We'll assume, first of all, that the rate at which we make deposits and the rate at which interest is compounded is the same; in other words, we won't make monthly deposits to an account that compounds weekly, for instance. If the compounding and the rate of deposit are both represented by n, we change this formula in the same way that we changed the compound interest formula to handle different compounding periods:

- Replace r, the annual interest rate, with $\frac{r}{n}$, splitting it into compounding periods.
- Replace t with nt to account for the interest compounding n times per year for t years.
- Rename P as PMT to emphasize that this is a regular payment.

This leads to the general formula:

$$\boxed{F = \frac{PMT\left[\left(1+\frac{r}{n}\right)^{nt} - 1\right]}{\left(\frac{r}{n}\right)}}$$

Payout Annuity Formula Suppose an account holds P dollars at the beginning, and this will be paid out in regular payments in the amount of PMT dollars each month (or other period, determined by n, but this won't affect this derivation).

This account is being drained by these payments, but at the same time, interest is accruing on the remaining balance. Thus, it turns out the future value that this account would have if it were allowed to grow by compound interest is equal to the future value it has as the reverse of a savings annuity (you can view it as the bank investing in you rather than you investing in the bank).

The future value from compound interest is

$$F = P\left(1+\frac{r}{n}\right)^{nt}$$

and the future value as a (reverse) savings annuity is

$$F = \frac{PMT\left[\left(1+\frac{r}{n}\right)^{nt}-1\right]}{\left(\frac{r}{n}\right)}$$

Thus, if we set these equal to each other, we get

$$P\left(1+\frac{r}{n}\right)^{nt} = \frac{PMT\left[\left(1+\frac{r}{n}\right)^{nt}-1\right]}{\left(\frac{r}{n}\right)}$$

and all we have to do is solve for P, the lump sum amount at the start of the payout process.

To do this, let's rewrite the right-hand side by separating out the denominator:

$$P\left(1+\frac{r}{n}\right)^{nt} = PMT\left(\frac{1}{\frac{r}{n}}\right)\left[\left(1+\frac{r}{n}\right)^{nt}-1\right]$$

Now, if we divide both sides by

$$\left(1+\frac{r}{n}\right)^{nt}$$

we only need to divide that through the last set of brackets on the right hand side:

$$P = PMT\left(\frac{1}{\frac{r}{n}}\right)\frac{\left[\left(1+\frac{r}{n}\right)^{nt}-1\right]}{\left(1+\frac{r}{n}\right)^{nt}}$$

Conveniently, that is the same as the first term in brackets, so that part of the division yields 1, and the rest becomes

$$\frac{1}{\left(1+\frac{r}{n}\right)^{nt}}.$$

Recall that we can use negative exponents to move terms from the denominator to the numerator, so

$$\frac{\left[\left(1+\frac{r}{n}\right)^{nt}-1\right]}{\left(1+\frac{r}{n}\right)^{nt}} = \left[1-\left(1+\frac{r}{n}\right)^{-nt}\right]$$

Putting the pieces back together gives the final formula:

$$\boxed{P = \frac{PMT\left[1-\left(1+\frac{r}{n}\right)^{-nt}\right]}{\left(\frac{r}{n}\right)}}$$

Exercises 1.4

In problems 1–3, a periodic deposit is made into an annuity with the given terms. Find how much the annuity will hold at the end of the specified amount of time.

1.

Regular deposit	\$250
Interest rate	4%
Frequency	Monthly
Time	15 years
Future value	?

2.

Regular deposit	\$10
Interest rate	5%
Frequency	Daily
Time	12 years
Future value	?

3.

Regular deposit	\$2000
Interest rate	3%
Frequency	Yearly
Time	22 years
Future value	?

In problems 4–6, find how much should be regularly deposited into an annuity with the given terms in order to have the specified final amount in the account.

4.

Regular deposit	?
Interest rate	5%
Frequency	Monthly
Time	18 years
Future value	\$50,000

5.

Regular deposit	?
Interest rate	6%
Frequency	Weekly
Time	10 years
Future value	\$27,000

6.

Regular deposit	?
Interest rate	3.5%
Frequency	Yearly
Time	35 years
Future value	\$200,000

In problems 7–9, you want to be able to withdraw the specified amount periodically from a payout annuity with the given terms. Find how much the account needs to hold to make this possible.

7.

Regular withdrawal	\$1000
Interest rate	5%
Frequency	Monthly
Time	20 years
Account balance	?

8.

Regular withdrawal	\$200
Interest rate	3%
Frequency	Weekly
Time	15 years
Account balance	?

9.

Regular withdrawal	\$20,000
Interest rate	5.5%
Frequency	Yearly
Time	25 years
Account balance	?

In problems 10–12, you expect to have the given amount in an account with the given terms. Find how much you can withdraw periodically in order to make the account last the specified amount of time.

10.

Regular withdrawal	?
Interest rate	4%
Frequency	Monthly
Time	18 years
Account balance	\$300,000

11.

Regular withdrawal	?
Interest rate	5%
Frequency	Weekly
Time	20 years
Account balance	\$250,000

12.

Regular withdrawal	?
Interest rate	2.85%
Frequency	Monthly
Time	30 years
Account balance	\$1,000,000

13. You deposit \$200 each month into an account earning 3% interest compounded monthly.

(a) How much will you have in the account in 30 years?

(b) How much total money will you put into the account?

(c) How much total interest will you earn?

14. You deposit \$1000 each year into an account earning 8% interest compounded annually.

(a) How much will you have in the account in 10 years?

(b) How much total money will you put into the account?

(c) How much total interest will you earn?

15. Evelyn has \$500,000 saved for retirement in an account earning 6% interest, compounded monthly. How much will she be able to withdraw each month if she wants to take withdrawals for 20 years?

16. Luke already knows that he will have \$750,000 when he retires. If he sets up a payout annuity for 30 years in an account paying 7% interest, how much could the annuity provide each month?

17. Michael is planning for retirement, and he estimates that he'll want to be able to withdraw \$2500 each month for 30 years once he retires. He opens a Roth IRA and finds investments that he expects to return 5% interest compounded monthly.

(a) How much will he need to have in the account when he retires in order to meet his goal?

(b) How much will he have to deposit each month for the next 40 years in order to get this balance at retirement?

(c) How much interest will his deposits earn?

18. Rachel is planning for retirement, and she estimates that she'll want to be able to withdraw \$1800 each month for 25 years once she retires. She opens a Roth IRA and finds investments that she expects to return 3.75% interest compounded monthly.

(a) How much will she need to have in the account when she retires in order to meet her goal?

(b) How much will she have to deposit each month for the next 40 years in order to get this balance at retirement?

(c) How much interest will her deposits earn?

19. Faith is 27 years old, she plans to retire at age 65, and she expects to live to age 92. She expects that her investments can earn an average return of 8% until retirement, and after retirement, she plans to earn 4%. If she wants to be able to withdraw \$2500 per month after retirement, how much should she start saving each month?

20. Caleb is 30 years old, he plans to retire at age 68, and he expects to live to age 90. He expects that his investments can earn an average return of 6% until retirement, and after retirement, he plans to earn 5%. If he wants to be able to withdraw \$2000 per month after retirement, how much should he start saving each month?

SECTION 1.5 Mortgages and Credit Cards

When the time comes to buy a home, you'll need to start shopping around for a **mortgage**, which is a long-term loan that is secured against whatever home you purchase. What this means is that if you fail to make your payments and *default* on your loan, the bank can *foreclose* on your house and take it from you in order to pay your debt. In this section you'll learn how to plan for buying a home.

Mortgage Concepts

Before we start doing calculations with specific formulas, we should discuss some of the terminology related to mortgages. There are plenty of terms we won't discuss here, but this should be enough to get started.

First, a mortgage is an example of an **installment loan**, which is a loan that is paid off in installments, meaning that you receive a lump sum when you take out the loan, and you pay this loan back in regular monthly payments. Installment loans are not unique to mortgages; this also describes car payments, for instance, so we'll start with examples about buying a car.

Next, you cannot borrow the full amount needed to purchase a home; for a large loan like this, you need to pay a **down payment** up front. The reason for this is simple: if none of your own money is tied up in the home, the bank is the one taking all the risk, so you need to come up with part of the money, and from their perspective this investment is your incentive to not default on the loan. Typical down payments range from 10% to 20% of the price of the home. There are exceptions, like FHA loans, which are guaranteed by the Federal Housing Administration; with these, you can put down as little as 3.5%. As we will see, though, it is in your best interest to save up as much as you can for the down payment, especially since a down payment lower than 20% requires **mortgage insurance**, which is a fee added to your monthly payment that pays for insurance against the bank's investment. This insurance will automatically drop off eventually, once you own at least 20% of the home.

The life of a mortgage can vary, but the most typical mortgages are structured for 30 years. Fifteen-year and 20-year mortgages are also fairly common; the faster you pay back the loan, the less you'll end up paying overall, but the trade-off is that your monthly payments will be higher (we'll see examples of this in this section). Mortgages can be **fixed-rate** or **adjustable-rate**. As the name implies, the interest rate can change for an adjustable-rate mortgage; the bank will recalculate the rate based on the current market. You may hear of a mortgage like a 5/1 ARM; this is an adjustable-rate mortgage (ARM) in which the rate is fixed for the first 5 years, and then every year after that, it is recalculated.

There are other costs in addition to simply paying back the loan, and we'll account for a few of them in this section. For instance, when you initiate a mortgage, there are one-time **closing costs**, which include things like taxes, fees to the lender and title company, land surveys, inspections, and so on. This just means that when you get ready to buy a home, you need to plan to have your down payment plus the closing costs on hand.

In addition to the loan payment (often referred to as *principal and interest* or *P&I*), a few other payments are added each month; the two most common are property taxes and homeowner's insurance. This is largely for convenience; rather than having to plan for a large insurance or tax payment at the end of the year, those get bundled into your mortgage payment so that you can easily budget for it all each month. The way this works is that the bank handles making the tax and insurance payments at the end of the year for you. Throughout the year, as the bank collects your mortgage payment, they set aside the portion of it that corresponds to taxes and insurance and place that into what is called an **escrow** account. Then at the end of the year, the bank automatically withdraws the right amount from the escrow account and pays the bills.

Summary: Mortgage Terminology

Installment Loan: A loan that is paid off in regular (usually monthly) payments.

Down Payment: A portion of the cost of a home (or car, etc.) that is paid up front; the loan covers the rest of the cost.

Mortgage Insurance: If a homebuyer puts down less than 20% of the cost, they must purchase insurance against defaulting.

Closing Costs: One-time fees that are paid at the time of purchase; these do not go toward the cost of the home.

Escrow: An account used to hold the portion of the mortgage payment designated for taxes and insurance.

That's a lot of information all at once, but it's important to understand these terms, because we will use them in the examples in this section.

Installment Loan Formula

At the core of the problems we'll be doing a bit later is the calculation of the loan payment, which is sometimes called P&I. This means that this payment is used to slowly pay down the principal of the loan, but it also must pay for the interest that the loan is accruing even as it is being paid off.

This may sound complicated now, but toward the end of the section, we'll break down exactly how a payment is divided between principal and interest, using *amortization tables*.

Calculating the payment on an installment loan: It turns out that we've actually seen this formula before: it's the one for finding the payment amount for a payout annuity.

To see this, imagine that you had \$5000 invested at a bank and you started taking out payments while earning interest (a payout annuity), and after 5 years your balance was zero. Now flip that around and put yourself in the position of a bank: now the lender is investing \$5000 in you (you take a loan of \$5000) and you start to pay them back in equal payments as the remaining balance earns interest, and after 5 years the balance is zero.

Since we're generally interested in finding the payment amount (we usually start by knowing how much we need to borrow), we'll use the version of the payout annuity formula that is solved for PMT, but if you want to start with what monthly payment you can afford and find out how much you can borrow, you can rearrange this formula, or simply refer to the version that we mainly used with payout annuities.

Installment Loan Payment

If an installment loan of P is taken out at an interest rate of r compounded n times a year, and paid back in equal payments n times a year over t years, the payment amount PMT is given by

$$PMT = \frac{P\left(\frac{r}{n}\right)}{1-\left(1+\frac{r}{n}\right)^{-nt}}$$

Let's do a few examples, starting with car loans, since that way we don't have to worry about escrow accounts and closing costs just yet.

EXAMPLE 1 CAR LOAN

If you take out an auto loan of \$11,000 at 4% interest for 60 months, what will your monthly payment be?

Solution Notice that we haven't mentioned a down payment here, but you are probably (hopefully) not borrowing the full cost of the car. Many car dealers will offer 0% down loans, but it is in your best interest to only borrow what you have to, not as much as they will offer you.

Start by organizing the given information:

P	\$11,000	The loan amount
r	0.04	4% interest rate
n	12	Payments made monthly
t	5	60 months = 5 years

Simply use the formula:

$$\begin{aligned} PMT &= \frac{\$11,000\left(\frac{0.04}{12}\right)}{1-\left(1+\frac{0.04}{12}\right)^{-(12)(5)}} \\ &= \boxed{\$202.58} \end{aligned}$$

Thus you'll end up paying \$202.58 every month for five years.

TRY IT If you take out an auto loan of \$15,000 at 5.6% interest for 72 months, what will your monthly payment be?

Now let's flip the question around: if you can budget for a car payment, how much of a loan can you afford?

Note: there's a common pitfall here. Depending on what salesperson you talk to, they may attempt to start with this question. The reason is that it's easy for you to imagine spending \$50 more each month, but if they told you how much of a total difference that makes up front, the number would be much more intimidating. You should always run your own calculations to check things like how much the total cost will change, and how much the total interest charge would change over time.

EXAMPLE 2

HOW MUCH CAN YOU AFFORD?

You can afford a monthly car payment of \$250. If you find a bank offering 4.85% interest on a 60 month loan, what is the largest car loan you can afford?

Solution

In this case PMT is known and P is the unknown that we want to find:

PMT	\$250	The loan amount
r	0.0485	4.85% interest rate
n	12	Payments made monthly
t	5	60 months = 5 years

Now use the formula and solve for P:

$$\$250 = \frac{P\left(\frac{0.0485}{12}\right)}{1-\left(1+\frac{0.0485}{12}\right)^{-(12)(5)}}$$

$$\$250 = P(0.0188)$$

$$\boxed{\$13,296} \approx P$$

You can afford a car loan of \$13,296 under these terms; of course, this is the maximum you can afford, so it wouldn't hurt to take out a smaller loan that this if you can.

TRY IT

You can afford a monthly car payment of \$300. If you find a bank offering 6.7% interest on a 48 month loan, what is the largest car loan you can afford?

EXAMPLE 3

ACTUAL COST

You see a TV ad that says "We can put you in the car of your dreams!!! Drive this brand-new \$25,000 car off the lot with only \$500 down and a monthly payment of \$550 for 60 months." How much do you end up actually paying for the car?

First, let's find out how much the car actually costs you.
The down payment is the amount that is due at the beginning, so if we add that to 60 payments of \$550, we'll have the total amount that comes out of your pocket:

Photo by Christopher Ziemnowicz

$$\$500 + (60)(\$550) = \$33,500$$

This means that you pay \$33,500 for a \$25,000 car, and the difference is the interest on the loan:

$$\$33,500 - \$25,000 = \boxed{\$8500}$$

Are you really willing to pay \$8500 in interest to have the car now, or can you save up first and pay for it in cash—at least in part—to reduce this interest cost?

TRY IT

A boat costs \$12,000, and you're offered a loan that requires \$1000 down and \$250 a month for 60 months. Find the total amount you would pay for the boat and the amount of interest you would pay with this loan.

This example emphasizes an important point: when you're offered a loan, don't focus on the monthly payment; instead, calculate the total cost of the loan and decide if it is worth it to you.

Mortgage Example

If you need to, you can review the terminology introduced at the beginning of the section, because we'll be using those terms in the following examples.

Fair warning: The next example is a long one, because it includes all the components we discussed at the beginning of the section. The good news is that this is a pretty realistic example, so if you can understand this, you'll be well prepared for the real thing.

EXAMPLE 4 BUYING A CONDO

The price of a condominium is \$180,000, and your bank offers a 30-year fixed mortgage at 4% interest. You have \$32,000 available right now.

(a) Your banker tells you to expect \$5000 in closing costs. What percentage down payment can you afford? Will you need mortgage insurance?

(b) What will the principal be on the mortgage?

(c) What will your monthly P&I payment be?

(d) In addition to principal and interest, your monthly payment will need to account for property taxes, homeowners insurance, and mortgage insurance, if necessary (based on part (a)):

Property Taxes:	1.5% of the home value per year
Homeowners Insurance:	\$900 per year
Mortgage Insurance:	\$40 per month

What will your total monthly payment amount be?

(e) How much will you pay in total over 30 years in principal and interest?

(f) How much interest will you pay in total?

Solution There's a lot here, but if we take it one part at a time, the actual calculations are fairly simple.

(a) Down payment and mortgage insurance:

Since you need to set aside \$5000 for closing costs, this leaves

$$\$32,000 - \$5000 = \$27,000$$

for a down payment. To find what percentage this represents of \$180,000, divide the down payment by the total cost:

$$\frac{\$27,000}{\$180,000} = 0.15 = 15\%$$

Therefore, you can afford a down payment of $\boxed{15\%}$, so **yes, you must pay for mortgage insurance**.

(b) Mortgage principal:

The principal on the mortgage is the amount you need to borrow, or the difference between the cost of the home and the down payment, the portion that you can pay in cash. Since the down payment will be \$27,000,

$$\begin{aligned}\text{Principal} &= \$180,000 - \$27,000\\ &= \boxed{\$153,000}\end{aligned}$$

(c) Monthly principal and interest payment (PMT):

This is where we actually have to pull out a complicated formula:

$$PMT = \frac{P\left(\frac{r}{n}\right)}{1 - \left(1 + \frac{r}{n}\right)^{-nt}}$$

We know that P, the principal, will be \$153,000, the interest rate is 4% (0.04), and t is 30 years, so we can calculate PMT:

$$\begin{aligned} PMT &= \frac{\$153,000\left(\frac{0.04}{12}\right)}{1-\left(1+\frac{0.04}{12}\right)^{-(12)(30)}} \\ &= \boxed{\$730.45} \end{aligned}$$

This is the amount we'll pay each month just on the loan, but the next part will add to it to yield our total payment amount, including the escrow and mortgage insurance (remember, we **do** require mortgage insurance in this example, but that will not be true in examples where the down payment is at least 20%).

(d) Total monthly payment:

Starting with the P&I amount of \$730.45, we also need to add taxes, homeowners insurance, and mortgage insurance. Start by calculating the monthly cost of each component (divide the yearly cost by 12):

P&I:		\$730.45 per month
Property Taxes:	Yearly: (1.5%)(\$180,000) = \$2700	\$225 per month
Home Insurance:	\$900 per year	\$75 per month
Mortgage Insurance:		\$40 per month

Adding them all together:

$$\begin{aligned} \text{Monthly Payment} &= \text{P\&I} + \text{Property Tax} \\ &\quad + \text{Home Insurance} + \text{Mortgage Insurance} \\ &= \$730.45 + \$225 + \$75 + \$40 \\ &= \boxed{\$1070.45} \end{aligned}$$

The final amount you can expect to pay each month is \$1070.45.

(e) Total principal and interest paid:

It's easy to calculate how much you will pay in total for the loan: simply multiply your monthly P&I payment by the number of payments you will make (12 payments a year for 30 years is (12)(30) = 360 payments). Note that the tax and insurance payments are not involved here; we're simply looking for the total principal and interest paid:

$$(\$730.45)(360) = \boxed{\$262,962}$$

Over 30 years, you'll pay a total of \$262,962 for this condo, including the loan interest.

(f) Total interest paid:

Now that we know how much you'll pay in principal *and* interest (from the last part), and we know what the principal on the loan is (part (b)), we can subtract to isolate the interest:

$$\$262,962 - \$153,000 = \boxed{\$109,962}$$

Notice that we didn't subtract the total cost of the condo, only the amount that we had to borrow from the bank (we don't pay interest on the amount covered by the down payment).

TRY IT

The price of a home is \$340,000, and your bank offers a 30-year mortgage at 3.5% interest. You have \$60,000 available right now, and the banker tells you to expect \$8000 in closing costs. Complete parts (a) - (f) of the previous example, using the same values for taxes and insurance.

Changing the Loan Terms

Let's take a look now at what happens if we change the terms of the loan. To begin, we'll set up a baseline using a 30-year fixed-rate loan at 4% and assume that we're borrowing $200,000. We won't need to account for a down payment, closing costs, or escrow, since we're only focused on the loan itself for the time being.

Then we'll see what happens if

(a) the interest rate changes,

(b) the loan amount changes (if, for instance, we made a larger down payment),

(c) or the length of the loan changes.

To make these comparisons, we'll use the same measurement each time: the total interest paid over the life of the loan. This is a good number to track, because it gives a sense of how much the loan costs (the principal must be paid back no matter what, but by altering the terms, we can alter the amount of interest).

We won't show the details of the calculations here, in order to focus on the results. If you like, you can refer to the description of using the TVM solver with payout annuities to follow along with the calculations quickly.

EXAMPLE 5 CHANGING THE INTEREST RATE

Compare a 30-year fixed-rate loan at 4% for $200,000 to the same loan at 3.5% by finding the total amount paid in interest for both versions.

Solution

To compare the total interest paid in each case, we need to find the monthly payment for each, then multiply this by the number of payments (360 for a 30-year loan) and subtract the principal of the loan ($200,000):

	4%	**3.5%**
Monthly payment:	$954.83	$898.09
Total paid:	$343,738.80	$323,312.40
Interest paid:	$\boxed{\$143,738.80}$	$\boxed{\$123,312.40}$

By reducing the interest rate by just half of one percentage point, you could save over $20,000 over time (not to mention that if you invested the $56.74 that you save each month at 7%, you could save up nearly $70,000 over the same time).

EXAMPLE 6 CHANGING THE LOAN AMOUNT

Compare a 30-year fixed-rate loan at 4% for $200,000 to the same loan for $180,000 by finding the total amount paid in interest for both versions.

Solution

We can copy the values from the previous example for the baseline loan, so we only need to calculate the monthly payment and other values for the reduced loan:

	$200,000	**$180,000**
Monthly payment:	$954.83	$859.35
Total paid:	$343,738.80	$309,366.00
Interest paid:	$\boxed{\$143,738.80}$	$\boxed{\$129,366.00}$

In this case, you could save over $34,000 in total by paying $20,000 more up front or finding a less-expensive home (or some combination of the two). Of that, about $14,000 is the amount saved on interest alone.

CHANGING THE LENGTH OF THE LOAN — EXAMPLE 7

Compare a 30-year fixed-rate loan at 4% for $200,000 to the same loan for 20 years and for 15 years by finding the total amount paid in interest for all three versions.

Solution

Again, the baseline loan will have the same values as always, and we will only change the length of the loan in the other two cases (note that to calculate the total paid, we multiply the 20-year monthly payment by 240, and we multiply by 180 for the 15-year loan):

	30 years	**20 years**	**15 years**
Monthly payment:	$954.83	$1211.96	$1479.38
Total paid:	$343,738.80	$290,870.40	$266,288.40
Interest paid:	$\boxed{\$143,738.80}$	$\boxed{\$90,870.40}$	$\boxed{\$66,288.40}$

Notice the trade-off here: shorter loans require higher monthly payments, because you have to pay down the loan faster, but the end result is that you pay much less in interest overall (a total of over $77,000 in the most extreme case). Of course, this all depends on the budget that you can work with.

You could, of course, experiment with changing all three factors at once, but at least these examples give us a sense of the total cost of a loan can change dramatically by adjusting the terms of the loan. When you start shopping for a mortgage, these are all factors to keep in mind.

Amortization Tables

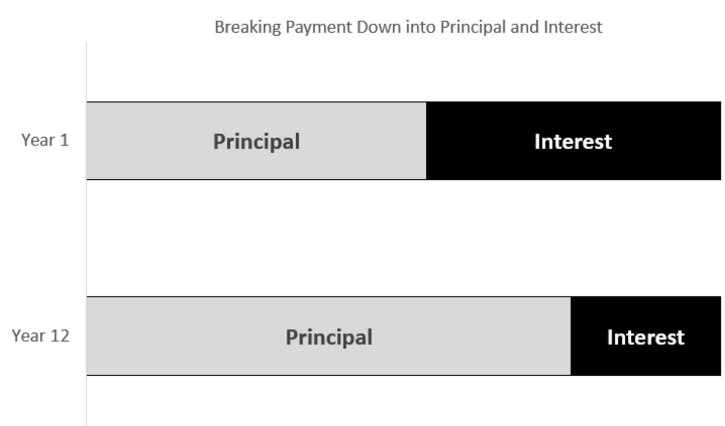
When you take out a mortgage, you'll receive in your loan packet a long table with hundreds of rows, and it's easy for your eyes to glaze over as you scan the columns of numbers marching down, but we're going to see that an **amortization table** is actually quite simple to understand.

The concept is this: we've used the term P&I (principal and interest) to describe the payment you make to the bank to pay back the loan. Each month, the balance that you currently own accrues interest, and part of your payment goes to pay off that interest. Whatever is left over after paying interest is used to pay down the balance.

This means that as time goes on, and the balance slowly drops, the amount you owe in interest each month decreases as well, so more of your monthly payment can be used to pay down the balance, driving it down even faster, and the process accelerates.

In practice, you'll notice that at the beginning of the process, a lot of your payment goes toward interest, but later in the life of the loan, the balance will shift, and the majority of your payment will go toward principal. The formula used to calculate your monthly payment ensures that at the end of the process, the balance drops to 0 exactly.

Here's a visual of what it looks like:



In year 1, the payment is about evenly split between principal and interest, but by year 12, the overwhelming majority goes to principal (this example was taken from a 20-year mortgage).

Notice that the overall payment never changes; it's just that the line separating the principal from the interest shifts slowly to the right over time. The final payment of the loan will include just pennies for interest.

Amortization Table

An **amortization table** lists the payments of an installment loan in order, showing the amount of each payment that goes toward interest and the amount that goes to toward principal.

An amortization table will typically have four columns: the payment number, the interest for that payment, the amount of that payment that goes toward principal, and the remaining balance after the payment.

Payment Number	Interest Payment	Principal Payment	Loan Balance

The calculations needed to build an amortization table are very simple, but repetitive, so we'll only build a few rows by hand, and we'll use Excel if we want a full table.

After building a table, you should notice that as you scan down the table, the interest payments decrease, the principal payments increase, and the balance slowly drops.

Let's see an example of building a table like this.

EXAMPLE 8 LOAN AMORTIZATION SCHEDULE

Suppose you take out a 20-year mortgage for \$200,000 at 7% interest, with monthly payments of \$1550.60 (we know how to calculate this now, but it is given to us to simplify this example). Prepare an amortization schedule for this loan.

Solution Only one calculation is really needed at each stage—calculating the interest due for that month (everything else follows from that). To calculate the interest due for a particular month, use the simple interest formula ($I = Prt$); since we're only looking at one payment period, there's no compounding happening. The principal P will be the loan balance at that point, r is the same for every payment, and t will be 1/12, since we're dealing with a month, a twelfth of a year.

1. The first payment:

$$\begin{aligned}
\text{Interest } &= Prt = (\$200,000)(0.07)\left(\frac{1}{12}\right) = \$1166.67 \\
\text{Principal Payment } &= \text{ Monthly Payment } - \text{ Interest} \\
&= \$1550.60 - \$1166.67 = \$383.93 \\
\text{Balance } &= \text{ Previous Balance } - \text{ Principal Payment} \\
&= \$200,000 - \$383.93 = \$199,616.07
\end{aligned}$$

2. The second payment: the starting balance for the second month is the final balance at the end of the first month, \$199,616.07.

$$\begin{aligned}
\text{Interest } &= Prt = (\$199,616.07)(0.07)\left(\frac{1}{12}\right) = \$1164.43 \\
\text{Principal Payment } &= \$1550.60 - \$1164.43 = \$386.17 \\
\text{Balance } &= \$199,616.07 - \$386.17 = \$199,229.90
\end{aligned}$$

To fill out the rest of the table, we could continue these calculations until we've covered all 240 payments, but of course this is far too tedious to do by hand, so we have a computer do it for us. The table below shows a few of the payments, skipping through to show payments at various stages of the loan.

Payment Number	Interest Payment	Principal Payment	Balance of Loan
1	1166.67	383.93	199616.07
2	1164.43	386.17	199229.90
3	1162.17	388.42	198841.47
4	1159.91	390.69	198450.79
⋮	⋮	⋮	⋮
30	1096.12	454.47	187452.64
31	1093.47	457.12	186995.52
⋮	⋮	⋮	⋮
145	663.44	887.16	112845.43
146	658.26	892.33	111953.09
⋮	⋮	⋮	⋮
239	17.93	1532.66	1541.61
240	8.99	1541.61	0.00

This illustrates the key features of an amortization table:

- The interest payment and principal payment in each row add up to the same monthly payment.
- The balance of the loan slowly shrinks and goes exactly to zero with the last payment.
- The amount of the payment that goes to interest shrinks each month and the amount that goes to paying down the principal grows by an equal amount.

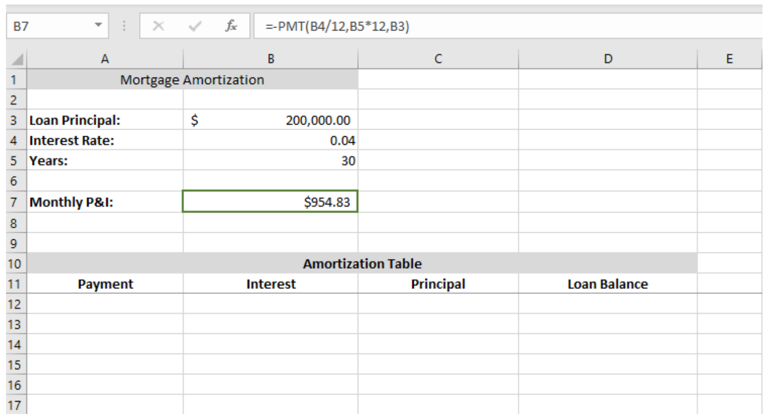
TRY IT

If you take out a loan for \$175,000 at 4.5% interest for 30 years, with a monthly payment of \$886.70, find values for A-F that will correctly fill out the first two rows of the amortization table below.

Payment Number	Interest Payment	Principal Payment	Balance of Loan
1	A	B	C
2	D	E	F

Using Excel to Create Amortization Tables

Again, we would never build a full amortization table by hand, but we can use a spreadsheet program to simplify the process. To begin, open Excel and enter the details of a loan, as shown. If we use these values to calculate the rest of the table, we can change any of the terms of the loan and watch the table update immediately for easy comparison.

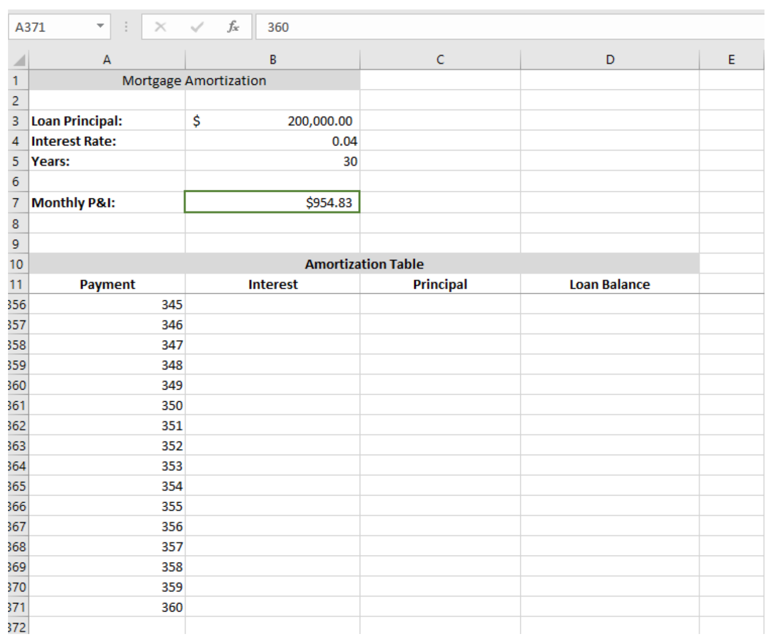


B7 | =-PMT(B4/12,B5*12,B3)

	A	B	C	D	E
1	Mortgage Amortization				
2					
3	Loan Principal:	$ 200,000.00			
4	Interest Rate:	0.04			
5	Years:	30			
6					
7	Monthly P&I:	$954.83			
8					
9					
10	Amortization Table				
11	Payment	Interest	Principal	Loan Balance	
12					
13					
14					
15					
16					
17					

Notice that to calculate the monthly payment, we're using the formula for PMT, as shown in the section on Saving for Retirement.

The key to avoid lots of typing is that you can select a cell or group of cells and drag downward from there, and Excel will try to interpret the pattern that you're trying to express. For instance, if we place a 1 in cell A12 (the first payment number), and a 2 below it, and we select those two and drag downward, the program will recognize that we want the payment number to grow by one each time, and it will fill in the progression (to drag down and repeat a pattern, move the cursor to the lower right corner of the selection, then click and drag):

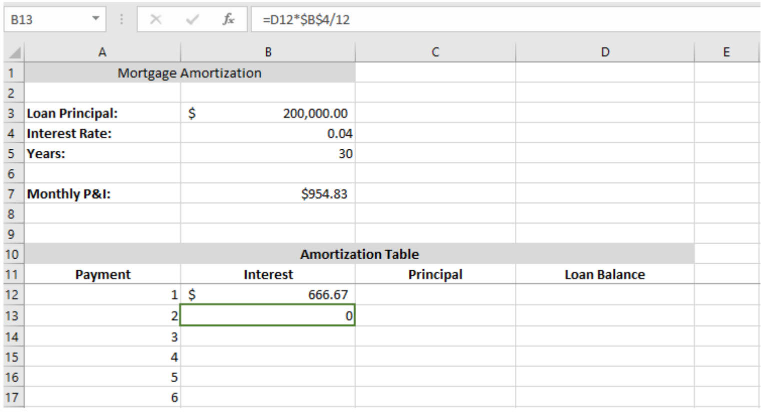


A371 | 360

	A	B	C	D	E
1	Mortgage Amortization				
2					
3	Loan Principal:	$ 200,000.00			
4	Interest Rate:	0.04			
5	Years:	30			
6					
7	Monthly P&I:	$954.83			
8					
9					
10	Amortization Table				
11	Payment	Interest	Principal	Loan Balance	
356	345				
357	346				
358	347				
359	348				
360	349				
361	350				
362	351				
363	352				
364	353				
365	354				
366	355				
367	356				
368	357				
369	358				
370	359				
371	360				
372					

We'll carefully fill in the first two rows of the table, then use this functionality to drag the formulas down and generate the rest.

The interest for each month will be calculated by multiplying the balance at the end of the previous month by the interest rate divided by 12. For the first row, we'll reference the starting balance ($200,000) but after that, we want this calculation to reference the entry in column D and the previous row.

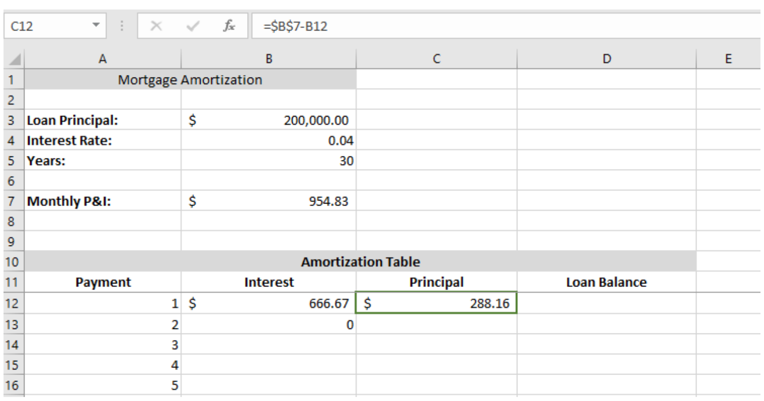


B13 =D12*$B$4/12

	A	B	C	D	E
1	Mortgage Amortization				
2					
3	Loan Principal:	$ 200,000.00			
4	Interest Rate:	0.04			
5	Years:	30			
6					
7	Monthly P&I:	$954.83			
8					
9					
10	Amortization Table				
11	Payment	Interest	Principal	Loan Balance	
12	1	$ 666.67			
13	2	0			
14	3				
15	4				
16	5				
17	6				

What's with the dollar signs in B4? Notice that instead of using B4 in the interest calculation, we are using $B$4. The reason for this is that as we drag our formula down, Excel assumes that we want all of the cells in our formula to move in the same way. This is, in fact, what we want to happen with the D12 in the formula; the next row should use D13, and so on. To tell Excel to keep using the value in B4 even as we drag the formula down, you need to use these dollar signs. You can insert them manually, or press F4 after clicking on the cell number in the formula.

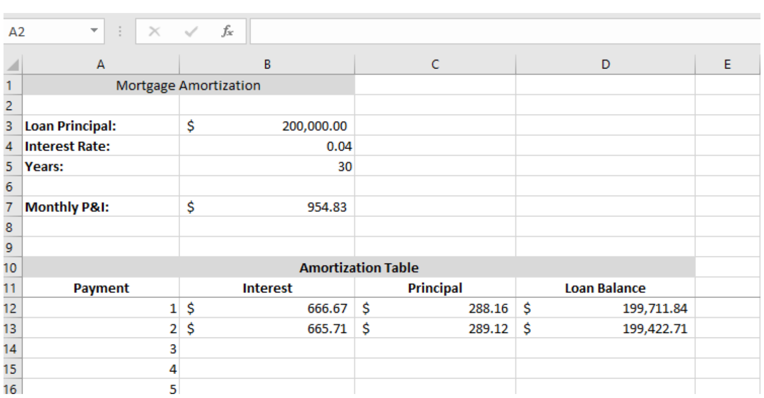
The last two calculations are the principal and the loan balance: the principal payment will be the total monthly payment ($B$7) minus the interest payment, and the loan balance will be the previous balance minus the principal payment for the current month.



C12 =$B$7-B12

	A	B	C	D	E
1	Mortgage Amortization				
2					
3	Loan Principal:	$ 200,000.00			
4	Interest Rate:	0.04			
5	Years:	30			
6					
7	Monthly P&I:	$ 954.83			
8					
9					
10	Amortization Table				
11	Payment	Interest	Principal	Loan Balance	
12	1	$ 666.67	$ 288.16		
13	2	0			
14	3				
15	4				
16	5				

After filling out the first two rows, you should have the following. Notice that we did the first two rows manually because the first row uses the starting balance several times, which is not consistent with the formula we want to copy.



A2

	A	B	C	D	E
1	Mortgage Amortization				
2					
3	Loan Principal:	$ 200,000.00			
4	Interest Rate:	0.04			
5	Years:	30			
6					
7	Monthly P&I:	$ 954.83			
8					
9					
10	Amortization Table				
11	Payment	Interest	Principal	Loan Balance	
12	1	$ 666.67	$ 288.16	$ 199,711.84	
13	2	$ 665.71	$ 289.12	$ 199,422.71	
14	3				
15	4				
16	5				

The formulas we want to copy down are the ones in the second row (cells B13 through D13), so if we select those cells and drag downward all the way to payment 360, we'll get the full table:

D371 =D370-C371

	A	B	C	D	E
1	Mortgage Amortization				
2					
3	**Loan Principal:**	$ 200,000.00			
4	**Interest Rate:**	0.04			
5	**Years:**	30			
6					
7	**Monthly P&I:**	$ 954.83			
8					
9					
10		**Amortization Table**			
11	**Payment**	**Interest**	**Principal**	**Loan Balance**	
350	339	$ 67.41	$ 887.42	$ 19,334.64	
351	340	$ 64.45	$ 890.38	$ 18,444.26	
352	341	$ 61.48	$ 893.35	$ 17,550.91	
353	342	$ 58.50	$ 896.33	$ 16,654.58	
354	343	$ 55.52	$ 899.32	$ 15,755.27	
355	344	$ 52.52	$ 902.31	$ 14,852.95	
356	345	$ 49.51	$ 905.32	$ 13,947.63	
357	346	$ 46.49	$ 908.34	$ 13,039.30	
358	347	$ 43.46	$ 911.37	$ 12,127.93	
359	348	$ 40.43	$ 914.40	$ 11,213.53	
360	349	$ 37.38	$ 917.45	$ 10,296.07	
361	350	$ 34.32	$ 920.51	$ 9,375.56	
362	351	$ 31.25	$ 923.58	$ 8,451.98	
363	352	$ 28.17	$ 926.66	$ 7,525.33	
364	353	$ 25.08	$ 929.75	$ 6,595.58	
365	354	$ 21.99	$ 932.85	$ 5,662.74	
366	355	$ 18.88	$ 935.95	$ 4,726.78	
367	356	$ 15.76	$ 939.07	$ 3,787.71	
368	357	$ 12.63	$ 942.20	$ 2,845.50	
369	358	$ 9.49	$ 945.35	$ 1,900.16	
370	359	$ 6.33	$ 948.50	$ 951.66	
371	360	$ 3.17	$ 951.66	$ (0.00)	
372					

This shows the end of the table, where the loan balance finally drops to \$0.00.

Credit Cards

So far, we've dealt with *fixed installment loans*, meaning that a specified amount is loaned and paid back with fixed payments in such a way that the balance goes to zero with the final scheduled payment.

On the other hand, there are **open-ended installment loans**, which require a variable payment each month, and the loan has no guaranteed end date; payments are made for as long as necessary to pay off the loan. The most common example is a credit card, where the total balance does not have to be paid off each month, and any unpaid balance rolls over to the following month. Of course, credit card companies take advantage of the ease of payment to rack up huge interest charges—credit card interest rates are among the highest you'll likely see. If, on the other hand, you pay off the entire balance each month, treating the card more like a debit card, you'll never pay any interest charges to your credit card company.

Average Daily Balance Method Different credit card companies calculate interest in different ways, all using the simple interest formula ($I = Prt$). The difference lies in how P is calculated; since the balance is constantly changing all month, they need a way to combine this all into a single principal. The method we'll illustrate is called the *average daily balance method*, which as the name suggests, takes the average of the balance on each day of the month. Thus, if the balance was \$100 on the first 15 days of the month and \$200 on the last 15 days, the average daily balance will be \$150.

To find the average daily balance, add up the balance for each day and divide by the number of days. In practice, we'll use a table to simplify the calculations by multiplying each different balance by the number of days that the card carried that balance.

CREDIT CARD CHARGES — EXAMPLE 9

Suppose your VISA card calculates interest using the average daily balance method, and the monthly interest rate is 1.5% (notice that this means that the nominal annual rate is $1.5\% \times 12 = 18\%$, which is not at all unusually high for a credit card). The itemized billing for the month of December is shown below.

Detail	Date	Amount
Unpaid balance	December 1	$1500
Payment received	December 4	$300
Groceries	December 8	$125
Gas	December 15	$45
Wendy's	December 22	$8.50
Last day of billing period	December 31	
Payment Due Date	January 7	

(a) Find the average daily balance.

(b) Find the interest due for December.

(c) Find the total balance owed on the last day of the billing period.

(d) This credit card requires a $15 minimum monthly payment or 1/36 of the amount due, whichever is higher. What is the minimum monthly payment due by January 7?

Solution

(a) Find the average daily balance.

To do this, we'll build a table to keep track of the unpaid balance after each transaction, and how long that unpaid balance lasts.

Date	Unpaid Balance
December 1	$1500
December 4	$1200
December 8	$1325
December 15	$1370
December 22	$1378.50

Now calculate how many days each balance lasted and multiply the balance by the number of days it lasted; this lets us quickly add up the balance for each day so that we can find the average by dividing this by the number of days.

Date	Unpaid Balance	Number of Days	(Unpaid balance) × (Number of Days)
December 1	$1500	3	$4500
December 4	$1200	4	$4800
December 8	$1325	7	$9275
December 15	$1370	7	$9590
December 22	$1378.50	10	$13,785
Total:		31	$41,950

The average daily balance is then the sum of the daily balances divided by 31, the number of days in the billing period:

$$\frac{\$41,950}{31} = \boxed{\$1353.23}$$

(b) Find the interest due for December.

Use the simple interest formula, noting that since the interest rate is given as a *monthly* rate, $t = 1$ since we're dealing with a single month:

$$I = Prt = (\$1353.23)(0.015)(1) = \boxed{\$20.30}$$

(c) Find the total balance owed on the last day of the billing period.

This is the final balance plus the interest charges:

$$\$1378.50 + \$20.30 = \boxed{\$1398.80}$$

(d) This credit card requires a \$15 minimum monthly payment or 1/36 of the amount due, whichever is higher. What is the minimum monthly payment due by January 7?

Since 1.36 of the amount due is \$1398.80/36 = \$38.86, which is more than \$15, the minimum payment due will be $\boxed{\$38.86}$

TRY IT

Suppose your VISA card calculates interest using the average daily balance method, and the monthly interest rate is 1.8%. The itemized billing for the month of May is shown below.

Detail	Date	Amount
Unpaid balance	May 1	\$850
Payment received	May 5	\$200
Groceries	May 7	\$240
Gas	May 13	\$33
Jewelry Store	May 25	\$575
Last day of billing period	May 31	
Payment Due Date	June 7	

(a) Find the average daily balance.

(b) Find the interest due for this month.

(c) Find the total balance owed on the last day of the billing period.

(d) This credit card requires a \$20 minimum payment or 1/24 of the amount due, whichever is higher. What is the minimum monthly payment due for this month?

We'll finish this discussion by taking another look at the trap of the minimum payment; we'll use the numbers from the preceding example. A minimum payment of \$38.84 on a balance of \$1398.13 sounds pretty reasonable, but think about how long it would take to pay off this balance by only making the minimum payment each month (even without adding further charges), since the majority of the minimum payment will go toward interest.

Skipping over the details (this can be figured out using a simple spreadsheet), if you started with a balance of \$1398.13 and never added another charge, just making the minimum payment each month, it would take 123 months to pay it off, or over 10 years. In doing so, you would end up paying a total of \$2571.46, or nearly twice what you owed. The lesson is simple: pay off your credit card in full as much as possible, and don't live beyond your means in a way that requires the use of credit to get by.

Exercises 1.5

1. If you take out an auto loan of \$8500 at 5% interest for 48 months, what will your monthly payment be?

2. If you borrow \$13,000 to buy a boat, and the bank charges 7% interest for 72 months, how much will you have to pay each month?

3. Janine bought \$3000 of new furniture on credit. Because her credit score isn't very good, the store is charging her a fairly high interest rate on the loan: 16%. If she agreed to pay off the furniture over two years, how much will she have to pay each month?

4. Carly financed a new \$1200 television at 12% for 48 months. How much will she have to pay every month to pay this off?

5. If you want to buy a car, and you can afford a monthly payment of \$175, how large of a loan can you get at 4.8% interest over 60 months?

6. Mary is going to finance new office equipment at a 2% rate over a 4 year term. If she can afford monthly payments of \$100, how much can she pay for the new office equipment?

7. If you buy a \$33,000 car for \$1000 down and monthly payments of \$685 for 60 months, how much will you pay in total for the car?

8. A car costs \$27,000, and you're offered a loan that requires \$800 down and a monthly payment of \$575 for 60 months, how much will you pay in interest?

9. A car dealership offers a loan with 3.5% interest for 60 months, and you plan to purchase a car for \$18,000. You can afford a down payment of \$2500.

(a) What will your monthly payment be?

(b) How much will you pay in total for the car?

(c) How much will you pay in interest over the life of the loan?

10. You plan to purchase a \$21,000 car, and your bank offers you a loan at 4.5% interest for 48 months. You can afford a down payment of \$4000.

(a) What will your monthly payment be?

(b) How much will you pay in total for the car?

(c) How much will you pay in interest over the life of the loan?

11. You want to buy a \$200,000 home, and you have \$40,000 saved up. The bank offers a 30-year mortgage at 3.8% interest.

(a) If you expect to pay \$6000 in closing costs, what percentage down payment can you afford?

(b) If you put less than 20% down, you'll need to pay mortgage insurance. Will you require mortgage insurance?

(c) What will the principal be on the loan?

(d) What will your monthly P&I payment be?

(e) In addition to principal and interest, the property taxes will be 1.5% of the home value per year, homeowners insurance will be \$750 per year, and the mortgage insurance (if needed, according to part (b)) will be \$25 per month. What will your total monthly payment amount be?

(f) How much will you pay in total over 30 years in principal and interest?

(g) How much interest will you pay in total?

12. You want to buy a \$375,000 home, and you have \$84,000 saved up. The bank offers a 20-year mortgage at 3.2% interest.

(a) If you expect to pay \$8000 in closing costs, what percentage down payment can you afford?

(b) If you put less than 20% down, you'll need to pay mortgage insurance. Will you require mortgage insurance?

(c) What will the principal be on the loan?

(d) What will your monthly P&I payment be?

(e) In addition to principal and interest, the property taxes will be 1.5% of the home value per year, homeowners insurance will be \$825 per year, and the mortgage insurance (if needed, according to part (b)) will be \$30 per month. What will your total monthly payment amount be?

(f) How much will you pay in total over 20 years in principal and interest?

(g) How much interest will you pay in total?

13. You can afford a \$900 per month mortgage payment. You've found a 30-year loan at 4% interest.

(a) How big of a loan can you afford?

(b) How much total money will you pay the bank?

(c) How much of that money is interest?

14. You can afford a \$1790 per month mortgage payment. You've found a 15-year loan at 3.25% interest.

(a) How big of a loan can you afford?

(b) How much total money will you pay the bank?

(c) How much of that money is interest?

15. If the interest rate on a 30-year mortgage for $175,000 were changed from 3.8% to 3.1%, how much would you save over the life of the loan?

16. How much would you save (over the life of the loan) on a 20-year mortgage at 4.5% if you reduced the amount you borrowed from $300,000 to $260,000?

17. If you borrow $250,000, how much could you save over the life of the loan if you took out a 15-year mortgage at 4.5% instead of a 30-year mortgage at 4%?

18. Suppose you take out a $315,000 mortgage for 30 years at 4.5% interest.

(a) Find the monthly payment on this mortgage.

(b) Fill out the first two rows of the amortization schedule below.

Payment Number	Interest Payment	Principal Payment	Balance of Loan
1			
2			

19. Suppose you take out a $180,000 mortgage for 15 years at 3.7% interest.

(a) Find the monthly payment on this mortgage.

(b) Fill out the first two rows of the amortization schedule below.

Payment Number	Interest Payment	Principal Payment	Balance of Loan
1			
2			

20. Suppose your VISA card calculates interest using the average daily balance method, and the monthly interest rate is 1.4%. The itemized billing for the month of April is shown below.

Detail	Date	Amount
Unpaid balance	April 1	$1100
Payment received	April 3	$500
New computer	April 11	$750
Books	April 15	$65
Mattress	April 28	$600
Last day of billing period	April 30	
Payment Due Date	May 7	

(a) Find the average daily balance.

(b) Find the interest due for this month.

(c) Find the total balance owed on the last day of the billing period.

(d) This credit card requires a $20 minimum payment or 1/36 of the amount due, whichever is higher. What is the minimum monthly payment due for this month?

21. Suppose your MasterCard calculates interest using the average daily balance method, and the monthly interest rate is 2.1%. The itemized billing for the month of August is shown below.

Detail	Date	Amount
Unpaid balance	August 1	$300
Payment received	August 9	$100
Tuition	August 10	$4500
Textbooks	August 18	$350
Groceries	August 25	$180
Last day of billing period	August 31	
Payment Due Date	September 7	

(a) Find the average daily balance.

(b) Find the interest due for this month.

(c) Find the total balance owed on the last day of the billing period.

(d) This credit card requires a $15 minimum payment or 1/24 of the amount due, whichever is higher. What is the minimum monthly payment due for this month?

22. Project: Finding a Mortgage
You and your family are looking to move and are shopping for a house. Your job is to find a mortgage that you can afford. You may choose your family size—you can be married with kids, married without kids, or single. You may also pick anywhere in the country that you'd like to live, but you can only make the median income listed for the state you choose. If you are married, you can assume that both you and your spouse are working and each are paid the median income for that state.

(a) Decide where you want to live. Do some research and find the median income for that state, and decide whether you are single or married, and whether or not you have children.

(b) Search realtor.com or a similar website to find a house that fits your family's needs. Take note of the

- List price of the home.
- Property taxes listed under the "Property History" tab. If property taxes are not listed, estimate the annual property taxes as 2% of the purchase price.

(c) Estimate the down payment you can afford, and take note of the principal of the loan that you will need.

(d) Do some research to find current interest rates. Use bankrate.com or a similar website to find mortgage rates (make sure to find a mortgage without *points*). Find three options: mortgages for 30 years, 20 years, and 15 years.

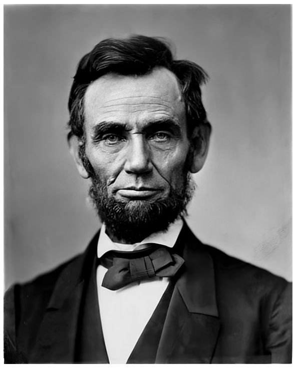
- What is the monthly payment for each option?
- How much will you pay in total for principal and interest over the life of the loan for each option?
- Keeping your monthly budget in mind, which option will you choose?

(e) Complete the following steps to find if you can afford this home. This worksheet uses a typical rule of thumb that you should not spend more than 28% of your income on housing.

Monthly Gross Income		
Borrower's annual income	\$______	
Co-borrower's annual income	+______	
Total gross annual income	\$______	
Divide total gross income by 12	÷ 12	
Total monthly gross income	\$______	
Find 28% of this	×0.28	
Allowable monthly housing cost	\$______	(A)
Monthly Taxes		
Home purchase price	\$______	
Estimated taxes	\$______	
Divide taxes by 12	÷12	
Monthly taxes	\$______	(B)
Monthly Housing Cost		
Monthly mortgage payment	\$______	+
Estimated monthly taxes (B)	\$______	+
Condo or homeowner's fee (if applicable)	\$______	
Total Monthly Housing Cost	= \$______	(C)

Compare (A) and (C). Can you afford the house you want to buy? If not, choose a less expensive house and redo this project.

SECTION 1.6 Income Tax



Before the Civil War, there was no federal income tax in the United States. At the beginning of the war, to help pay for it, Congress passed the Revenue Act of 1861, which imposed a tax of 3% on all incomes over $800. The following year, the Revenue Act of 1862 instituted a 3% tax on incomes over $600 and 5% on incomes over $10,000. These taxes were repealed after the war, and it wasn't until the ratification of the Sixteenth Amendment in 1913 that a federal income tax was finally established for good.

Interestingly, these two laws passed by Congress in 1861 and 1862 introduce us to one of the main distinctions between types of taxes; specifically, they give us examples of **flat** taxes and **progressive** taxes.

Flat taxes–like sales tax, property tax, or gasoline tax–are calculated as a single percentage of a total.

EXAMPLE 1 **PROPERTY TAXES**

Joan paid $3,200 in property taxes on her house, which was valued at $215,000 last year. What is the property tax rate?

Solution The equivalent percentage is

$$\frac{3,200}{215,000} = 0.01488 = \boxed{1.49\%}$$

In the section on Applied Percentage Problems, we worked out examples involving sales tax; all you need to know is the total charge before taxes and the tax rate, and you can calculate the sales tax by multiplying these.

Progressive taxes–like the current federal income tax–are calculated using different tax rates for different income levels. Specifically, the term *progressive* means that tax rates increase for higher income levels. A tax that did the opposite (lowered tax rates for higher incomes) would be called a *regressive tax*; these are not generally used in practice, but you may hear this term, for instance, to describe lottery tickets, since they are bought more frequently by people on the lower end of the income scale.

Tax Categories

ex: sales tax
- A **flat tax** charges a consistent percentage, no matter what the taxed amount is.

ex: income tax
- A **progressive tax** charges a higher percentage for higher taxed amounts.

The two bills passed by Congress during the Civil War are examples of these two categories. The tax created in 1861 was a flat tax, although technically it could be regarded as a progressive tax with a tax rate of 0% for all incomes under $800. But income tax was incredibly simple to calculate: simply subtract $800 from your annual income, and multiply the result by 3%.

The 1862 tax was a progressive tax with two different tax rates depending on income.

Progressive Taxes: Income Tax

This brings us to the key feature of the calculations we'll be doing in this calculation: **the progressive tax rates do not apply to *all* of your income**. Instead, a taxpayer's income is split into segments, and each segment is taxed at the rate that applies to it.

Let's use the 1862 tax to illustrate a simple example. Remember that incomes over $600 were taxed at 3% and incomes over $10,000 were taxed at 5%.

For comparison, consider three people:

1. Mary is a schoolteacher, earning $360 a year.
2. John is a schoolteacher, earning $846 a year.
3. William is a member of Congress and a businessman, with a total income of $12,000.

Now calculate how much each owes in taxes:

1. Since Mary makes less than \$600 a year, she owes no taxes.
2. Since John earns more than \$600, but less than \$10,000, only the first tax rate applies to him. However, *only his income over \$600* gets taxed. This means that he's only taxed on \$246, and 3% of \$246 is \$7.38, so that's John's full tax bill.
3. William makes more than \$10,000, so both tax rates apply to him. Just like John, his first \$600 are not taxed at all. At that point, his income is taxed at 3% all the way from dollar 600 up to dollar 10,000, and everything beyond that is taxed at 5%.

   It may be simplest to start at the top: he has \$2,000 past the \$10,000 threshold, so that amount is taxed at 5%: (5%)(\$2000) = \$100. Then from the \$10,000 mark down to the \$600 mark represents a total of

   $$\$10,000 - \$600 = \$9400$$

   which is taxed at 3%:

   $$(3\%)(\$9400) = \$282.$$

   William's total tax bill, then, is the sum of these two, for a total of \$382.

It can help to think of a person's income as if they are holding dollar bills, and placing their money into a series of buckets. The first bucket (in the example above) can hold \$600, so once that's been filled, the person starts putting their money in the next bucket, which can hold \$9400, and so on. Once the person has finished splitting their money this way, a certain percentage of each bucket is removed, and they can keep the rest.

If you've heard the term **tax brackets**, it refers to this progressive pattern. Usually, if someone mentions what tax bracket they belong to, they really mean the highest bucket that they reach, but crucially, *not all of their income is taxed at that rate.* Whether you make \$40,000 or \$4,000,000 a year, your first dollar will be taxed at the same rate.

Using Modern Tax Rates

Let's take the example of someone who makes \$82,350 this year (using 2020, the year of publication). In 2020, a single taxpayer's first \$9,875 are taxed at 10%, everything from that point to \$40,125 is taxed at 12%, and from that point to \$85,525 is taxed at 22%. We can summarize this with a table like the one below.

Tax Rate	**Income**
10%	up to \$9,875
12%	\$9,875 to \$40,125
22%	\$40,125 to \$85,525

There's more to this table, but that's as far as we need to go, since our fictional taxpayer doesn't make more than \$85,525.

\$82,350

Dollars	1 – 9,875	9,876 – 40,125	40,126 – 82,350	Split money into brackets
Tax Rate	10%	12%	22%	
Calculation	\$9,875(0.1)	(\$40,125 – \$9,875)(0.12)	(\$82,350 – \$40,125)(0.22)	Multiply the amount in each
	= \$987.50	= \$3,630.00	= \$9,289.50	bracket by the tax rate

This taxpayer pays $\$987.50 + \$3,630.00 + \$9,289.50 = \$13,907$ in taxes.

The tax rates for each bracket may change from year to year, but the process remains the same:

1. Divide the taxable income into these brackets, putting each dollar into the appropriate bracket until you get to the end of the income (so if the taxpayer's income in the example above was $32,000, we would have stopped in the second bracket and not spilled over into the third).
2. Multiply the amount in each bracket by the tax rate for that bracket.
3. Add up the tax amounts from each bracket to find the total income tax owed.

The following table is the tax table for 2020, showing the tax brackets and associated tax rate for each.

Tax Rate	Single or Married Filing Separately	Married Filing Jointly	Head of Household
10%	up to $9,875	up to $19,750	up to $14,100
12%	$9,875 to $40,125	$19,750 to $80,250	$14,100 to $53,700
22%	$40,125 to $85,525	$80,250 to $171,050	$53,700 to $85,500
24%	$85,525 to $163,300	$171,050 to $326,600	$85,500 to $163,300
32%	$163,300 to $207,350	$326,600 to $414,700	$163,300 to $207,350
35%	$207,350 to $518,400	$414,700 to $622,050	$207,350 to $518,400
37%	more than $518,400	more than $622,050	more than $518,400
Standard Deduction	$12,400	$24,800	$18,650

Note that there is an extra row at the bottom that describes the *standard deduction*; we'll discuss that shortly. Also, the term "Head of Household" may be confusing, but it simply means a single individual with one or more dependents.

EXAMPLE 2 INCOME TAX

Using the tax table above, how much would a married taxpayer who files separately from their spouse owe on a taxable income of $98,400?

Solution First, note that we'll be using the first column, since this taxpayer is married, filing separately. Next, divide the $98,400 into the appropriate brackets:

$98,400

Dollars	1 – 9,875	9,876 – 40,125	40,125 – 85,525	85,525 – 98,400
Tax Rate	10%	12%	22%	24%

Then simply multiply the amount in each bracket by that tax rate and add up these totals:

$$\begin{aligned}\text{Tax Owed} &= (9,875)(0.1) + (40,125 - 9,875)(0.12) \\ &\quad + (85,525 - 40,125)(0.22) + (98,400 - 85,525)(0.24) \\ &= \boxed{\$17,695.50}\end{aligned}$$

Notice that the *effective tax rate* for this taxpayer (the percentage of their taxable income that they actually paid) was

$$\frac{\$17,695.50}{\$98,400.00} = 0.1798$$

or about 18%.

Using the 2020 tax table, how much would a head of household owe on a taxable income of $47,000?

TRY IT

Deductions and Credits

We've used the term **taxable income** several times; what does that mean?

It turns out that you won't be taxed on your full income; by law, there are ways that you can reduce your tax burden, and broadly speaking, these fall into two categories: **deductions** and **credits**.

Warning: You should know that the rest of this discussion presents a simplified view; we're glossing over many of the fine details here. The goal of this section is simply to give you a broad understanding of how taxes are calculated.

Deductions and Credits

Tax deductions and credits are both designed to reduce the amount of income a taxpayer owes. The difference between them is when they are applied.

Tax Deductions are subtracted from the taxpayer's *gross income*, which is the total amount of income they receive in a year. By subtracting these, the taxable income is reduced, which ultimately results in a lower tax calculation.

Tax Credits are subtracted *after* calculating taxes; i.e. they are subtracted from the tax owed, as determined using the tax table.

Tax credits have a more direct impact on the tax owed, but deductions are more common.

Examples of Deductions and Credits: Common tax deductions include things like

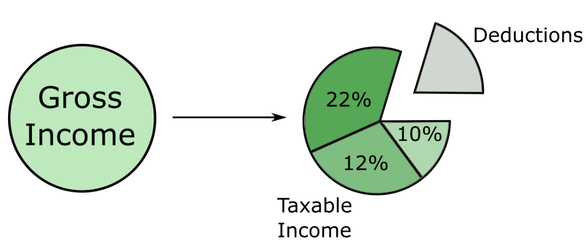
- Interest paid on a mortgage
- Contributions to a traditional IRA (since traditional IRAs are taxed in retirement)
- Charitable contributions
- Education expenses
- Self-employment expenses

Tax credits include

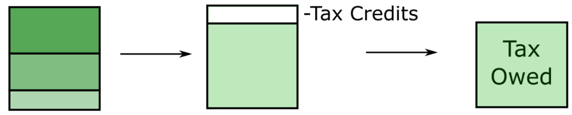
- The Lifetime Learning Credit, which rewards undergraduate and graduate study
- Child tax credit and adoption credit
- Earned income tax credit, for low- and moderate-income taxpayers

There are often short-term tax credits created to incentivize behavior like purchasing electric vehicles or energy-efficient appliances.

In visual terms, it all begins with the gross income, the total number for all money coming in from employment, investment returns, side jobs, and so on. The deductions are removed from this, and the remainder (the taxable income) is split into buckets, according to the tax brackets.



Once the appropriate percentage of each segment has been accounted for, this forms the initial tax calculation, but then any applicable tax credits are subtracted, resulting in the final tax owed.



For instance, if a taxpayer with a gross income of $50,000 could claim deductions totaling $13,000 and found $500 worth of tax credits, it would work out something like this:

- **Gross Income:** $50,000
- **Taxable Income:** (subtract the deductions) $50,000 - $13,000 = $37,000
- **Initial Tax:** (skipping the calculation, but use the tax table) $4242.50
- **Final Tax Owed:** (subtract the credits) $4242.50 - $500 = $3742.50

This person would owe a final tax bill of $3742.50.

That's the basics of deductions and credits, but there's one final piece to the puzzle that we need to understand: **itemized** deductions vs. the **standard** deduction.

Standard and Itemized Deductions

Taxpayers can choose to either take the standard deduction (listed on the tax table) or itemize their deductions (add up all the deductions that apply to them).

In practice, a taxpayer will add up the itemized deductions, and if this total is larger than the standard deduction, they will choose to go the itemized route. If not, they will choose to use the standard deduction.

Basically, there's a minimum deduction amount; if you find that your itemized deductions surpass that, you can use this larger sum as your deduction total.

With that, we're ready to work out a full example.

EXAMPLE 3 TAX CALCULATION

Use the 2020 tax table on page 72 to calculate the final tax owed by a single taxpayer whose details are given below.

Gross income:	$65,000
Deductions:	$3000: charitable donations
	$6000: contribution to traditional IRA
	$1500: education expenses
	$300: cost of tax preparation
Tax credit:	$500: energy-efficient appliances

Solution First, decide whether this person will use itemized deductions or the standard deduction:

Calculate deductions

$$\begin{aligned}\text{Itemized deductions} &= \$3000 + \$6000 + \$1500 + \$300 \\ &= \$10,800\end{aligned}$$

Since the standard deduction for a single taxpayer ($12,400) is larger, they will use the standard deduction.

Next, subtract the deductions from the gross income to find the taxable income:

Find the taxable income

$$\begin{aligned}\text{Taxable income} &= \text{Gross income} - \text{deductions (standard)} \\ &= \$65,000 - \$12,400 \\ &= \$52,600\end{aligned}$$

Now divide the taxable income into its brackets and calculate the tax owed from each bracket:

Divide the taxable income into brackets

$$\overbrace{\begin{array}{r|ccc} \text{Dollars} & 1 - 9{,}875 & 9{,}876 - 40{,}125 & 40{,}126 - 52{,}600 \\ \text{Tax Rate} & 10\% & 12\% & 22\% \end{array}}^{\$52{,}600}$$

Multiply the amount in each bracket by that tax rate

$$\begin{aligned} \text{Tax Owed} &= (9,875)(0.1) + (40,125 - 9,875)(0.12) \\ &\quad + (52,600 - 40,125)(0.22) \\ &= \$7,362 \end{aligned}$$

Finally, subtract the tax credit from this initial tax calculation to find the final amount that they owe:

Subtract the tax credit

$$\text{Final Tax Owed} = \$7,362 - \$500 = \boxed{\$6,862}$$

TRY IT

Use the 2020 tax table to calculate the tax owed by a married couple filing jointly whose details are given below.

Gross income:	$74,000
Deductions:	$5500: charitable donations
	$150: cost of tax preparation
Tax credit:	$700: child tax credit

Tax Withholding

During World War II, when the U.S. government was in dire need of tax revenue in order to avoid the rampant inflation that occurred during World War I, the IRS began to investigate tax withholding. Rather than collecting taxes at the end of the year, they began to withhold taxes from workers' paychecks throughout the year, and then make up the difference at the end, either by a refund or an extra tax payment. Milton Friedman, who won the 1976 Nobel Prize in economics, helped to formulate this plan.

New employees fill out a W-4 form, which is intended to calculate as accurately as possible how much they'll owe in taxes, so that the correct amount can be collected. Many people like the idea of overestimating, because if more is withheld throughout the year, they receive a huge refund check. However, this is not actually a good policy, since in effect they are giving the government an interest-free loan for the entire year with no strings attached.

Exercises 1.6

1. A ________________ is a tax at a consistent rate.

2. A ________________ is a tax for which the rate increases for higher taxed amounts

3. A ________________ is a tax for which the rate decreases for higher taxed amounts

4. If the property taxes are \$1800 on a home valued at \$140,000, what is the effective tax rate?

5. In state A, the gas tax is 28 cents per gallon, where the average pre-tax cost of gas is \$2.58 per gallon. In state B, the gas tax is 25 cents per gallon, where the average pre-tax cost of gas is \$2.50. Which state has a lower gas tax rate?

6. If the sales tax is \$16.05 on a purchase of \$214, what is the sales tax rate?

7. Using the 2020 tax table on page 72, how much would a single taxpayer owe on a taxable income of \$55,000?

8. Using the 2020 tax table on page 72, how much would a married couple filing jointly owe on a taxable income of \$92,000?

For problems 9–14, use the marginal tax table on page 72 to calculate the tax owed by each taxpayer.

9.

Taxpayer:	Single
Gross income:	\$75,000
Deductions:	\$18,000: mortgage interest
	\$2500: property taxes
	\$2000: charitable donations
	\$300: cost of tax preparation
Tax credit:	\$800

10.

Taxpayer:	Single
Gross income:	\$40,000
Deductions:	\$10,000: mortgage interest
	\$2000: property taxes
	\$300: charitable donations
Tax credit:	\$1300

11.

Taxpayer:	Married, filing jointly
Gross income:	\$85,500
Deductions:	\$5000: charitable donations
	\$3750: state taxes
Tax credit:	\$750

12.

Taxpayer:	Married, filing jointly
Gross income:	\$52,000
Deductions:	\$9000: mortgage interest
	\$4500: charitable donations
	\$1500: theft loss
	\$1800: state taxes
Tax credit:	\$1400

13.

Taxpayer:	Head of Household
Gross income:	\$104,000
Deductions:	\$18,000: mortgage interest
	\$5300: property taxes
	\$4800: state taxes
Tax credit:	none

14.

Taxpayer:	Head of Household
Gross income:	\$43,000
Deductions:	\$3700: property taxes
	\$3650: state taxes
Tax credit:	none

15. Project: Flat Tax, Modified Flat Tax, and Progressive Tax
Many people have proposed various revisions to the income tax collection in the U.S. Some, for example (including Milton Friedman), have claimed that a flat tax would be fairer. Others call for revisions to how different types of income are taxed. This project is for you to investigate this idea.

Imagine the country is made up of 100 households. The federal government needs to collect \$800,000 in income taxes to be able to function (this is roughly proportional to the actual U.S. population and tax needs, but using smaller, more manageable numbers). The population consists of 6 groups:

Group A:	20 households that earn \$12,000 each
Group B:	20 households that earn \$29,000 each
Group C:	20 households that earn \$50,000 each
Group D:	20 households that earn \$79,000 each
Group E:	15 households that earn \$129,000 each
Group F:	5 households that earn \$295,000 each

We are going to determine new income tax rates.

Proposal A The first proposal we'll consider is a flat tax — one where every income group is taxed at the same percentage rate.

1) Determine the total income for the population.

2) Determine what flat tax rate would be necessary to collect enough money.

Proposal B The second proposal we'll consider is a modified flat-tax plan, where everyone only pays taxes on any income over \$20,000. So everyone is Group A will pay no taxes, for instance, and everyone in Group B will pay taxes only on \$9,000.

3) Determine the total *taxable* income for the population.

4) Determine what flat tax rate would be necessary to collect enough money in this modified system.

5) Complete the table below for both plans.

		Flat Tax Plan		Modified Flat Tax Plan	
Group	Income per Household	Income Tax per Household	Income After Taxes	Income Tax per Household	Income After Taxes
A	$12,000				
B	$29,000				
C	$50,000				
D	$79,000				
E	$129,000				
F	$295,000				

Proposal C The third proposal we'll consider is a progressive tax, where lower income groups are taxed at a lower percentage rate and higher income groups are taxed at a higher percentage rate. For simplicity, we're going to assume that a household is taxed at the same rate on *all* their income.

6) Set progressive tax rates for each income group to bring in enough money. There is no single right answer here — just make sure you bring in enough money (the total tax must add up to at least $800,000)!

Group	Income per Household	Tax Rate (%)	Income Tax per Household	Total Tax Collected for All Households	Income After Taxes per Household
A	$12,000				
B	$29,000				
C	$50,000				
D	$79,000				
E	$129,000				
F	$295,000				

7) Discretionary income is the income people have left over after paying for necessities like rent, food, transportation, etc. The cost of basic expenses does increase with income, since housing and car costs are higher. However, these increases are usually not proportional to the increase in income. For each income group, estimate their essential expenses and calculate their discretionary income. Then compute the effective tax rate for each plan relative to discretionary income rather than income.

Group	Income per Household	Discretionary Income (estimated)	Effective Rate, Flat	Effective Rate, Modified	Effective Rate, Progressive
A	$12,000				
B	$29,000				
C	$50,000				
D	$79,000				
E	$129,000				
F	$295,000				

8) Which plan seems the most fair to you? Which plan seems the least fair to you? Why?